How do $L^2$ eigenfunctions decay on a quantum graph?
Main question

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Prior work:
- Liouville and Green for 1D, Agmon: negative energies
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- Liouville and Green for 1D, Agmon: negative energies
- Hislop-Post: Worked with "radial trees," i.e. identical lengths and identical branching numbers, and a potential added at the vertices as a vertex condition; positive energy localization
- Random lengths model
Agmon philosophy

For negative eigenvalues (actually $E < \lim \inf V$), use tricky integration by parts to show that $\psi \in L^2$ and $H\psi = E\psi \implies \exp(\rho(0, x))\psi \in L^2$, where $\rho$ is a metric. Once $L^2$ exponential decay is established, they can be used to establish pointwise estimates showing exponential decrease. The method works essentially if $x : V(x) - E \leq 0$ is compact, and the expected metric is essentially an “action integral” of $(V(x) - E)^{1/2}$ over the minimizing path.
Notations:

Our setup:

- Quantum graph $\Gamma$; edges are segments of the real line
- A function space $\mathcal{K}$ on $\Gamma$ so that for $\phi \in \mathcal{K}$
  - $\phi$ is twice differentiable on the edges
  - $\phi$ is continuous at the vertices

Notice: if $f, g \in \mathcal{K}$ then so is $fg$. 
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- Quantum graph $\Gamma$; edges are segments of the real line
- A function space $\mathcal{K}$ on $\Gamma$ so that for $\phi \in \mathcal{K}$
  1. $\phi$ is twice differentiable on the edges
  2. $\phi$ is continuous at the vertices
  3. If $v$ is a vertex, $e_j$ are the edges adjacent to $v$, and $\phi_{e_j}$ is $\phi$ restricted to the edge $e_j$, then $\sum \phi'_{e_j}(v) = 0$ where the derivative is taken in the direction away from the vertex $v$
  4. Whatever comes in must come out
  5. Kirchoff condition
  6. Notice: if $f, g \in \mathcal{K}$ then so is $fg$.
- Differential operator $H = -\Delta + V(x)$ acting on $\mathcal{K}$
Case study: the Line

Definition 1

Let $\rho_E(x, y) = \int_x^y \sqrt{(V(t) - E)_+} \, dt$.

Notes:

- In multiple dimensions or on graphs with non-trivial topology, take the minimum over all possible paths from $x$ to $y$. 
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Theorem 2 (Agmon)

Let \( H = -\Delta + V \) with \( V \) real and continuous be a closed operator bounded below with \( \sigma(H) \subset \mathbb{R} \). Suppose \( E \) is an eigenvalue of \( H \) and that \( \text{supp}(E - V(x))_+ \) is compact. Suppose \( \psi \in L^2 \) is an eigenfunction of \( H \). Then for any \( \epsilon > 0 \) there exists a constant \( c_\epsilon \) such that

\[
\int e^{2(1-\epsilon)\rho_E(x)} |\psi(x)|^2 \, dx \leq c_\epsilon
\]
Lemma 3

For $\phi \in L^2$ and $F_\alpha$ bounded and satisfying $V - E - \left| \frac{F'_\alpha}{F_\alpha} \right|^2 > \delta$, we get that

$$\langle F_\alpha \phi, (H - E) \frac{1}{F_\alpha} \phi \rangle \geq \delta \|\phi\|^2$$
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Proof:

$$\langle F_\alpha \phi, (H - E) \frac{1}{F_\alpha} \phi \rangle = \int (V - E)|\phi|^2 + (F_\alpha \phi)' \left( \frac{\phi}{F_\alpha} \right)' + BT$$

$$= \int (V - E)|\phi|^2 + |\phi'|^2 - \left| \frac{F'_\alpha \phi}{F_\alpha} \right|^2 \geq \delta \|\phi\|^2.$$

Dropped the term $|\phi'|^2$, since positive. Integrated by parts, since $\phi \in L^2$ and $F_\alpha$ is bounded. Note $\delta$ is independent of the upper bound on $F_\alpha$. 
Proof Ingredient # 2

Lemma 4

Let \( \eta \) be a smoothed characteristic function of \( \{ V - E > \delta \} \) such that \( \eta' \) has compact support and \( \psi \) be an \( L^2 \) solution of \((H - E)\psi = 0\). Then

\[
\langle F_\alpha^2 \eta \psi, (H - E)\eta \psi \rangle = \langle \eta' (\eta F_\alpha^2)' \psi, \psi \rangle \leq C \|\psi\|^2. \tag{3.1}
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Lemma 4

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- For the equality can use that $(H - E)\psi = 0$ and integrate by parts
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- Can construct such \( \eta \) because we assume \( E \) to be an eigenvalue below the spectrum, equivalent to \( V - E \leq \delta \) compact. Here \( \eta = 1 \) on the exterior set and 0 on the compact set
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- Can construct such $\eta$ because we assume $E$ to be an eigenvalue below the spectrum, equivalent to $V - E \leq \delta$ compact. Here $\eta = 1$ on the exterior set and 0 on the compact set.
- Here we have taken $\phi$ from before to be $\eta F_\alpha \psi$, so that $\delta \|\eta F_\alpha \psi\|^2 \leq C \|\psi\|^2$. 

Proof ingredient # 3

We want to take $F$ as large as possible, but still satisfying the constraint $V - E - \left| \frac{F'}{F} \right|^2 > \delta$, so we construct it by

$$F(x) = e^{(1-\delta) \int_0^x (V(t)-E)_+} = e^{(1-\delta) \rho_E(x)}$$

In above lemmas, we use $F_\alpha$ instead where for $\alpha > 0$

$$F_\alpha := \frac{F}{1 + \alpha F} < C_\alpha$$
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We observe that

- (i) $F_\alpha(x) < F(x)$,
- (ii) $|F'_\alpha(x)| < |F'(x)|$,
- (iii) $\left| \frac{F'_\alpha(x)}{F_\alpha(x)} \right| < \left| \frac{F'(x)}{F(x)} \right|$.

Then can take $\alpha \to 0$, since the constants throughout do not depend on $\alpha$. 
For a quantum tree:

- Same argument carries over to rooted trees with few modifications.
- Integration by parts still works because of the Kirchoff condition.
- Can show that if $\psi \in L^2(\Gamma)$ then $e^{(1-\epsilon)\rho_E(x)}\psi \in L^2$, where
  $$\rho = \int_0^x (V(t) - E)^{1/2}$$
  and the integral is taken over the unique path from the root to $x$.
- By a standard method can go from $L^2$ estimates to pointwise.
- Are we done?
Case study: the regular tree

- Regular tree: starts at a root, each edge has length $L$ and at each vertex splits into $b$ edges, e.g. the Bethe Lattice.
- Fix $E < 0$. Consider $H = -\Delta$, i.e. $V = 0$ and look for a $L^2$ solution.
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- Fix $E < 0$. Consider $H = -\Delta$, i.e. $V = 0$ and look for a $L^2$ solution.
- The problem: Does there exist an $L^2$ solution and does it decay faster than $e^{-\sqrt{|E|}x}$, which would be the corresponding decay on the line.
Construction for the Regular Tree

Transfer matrix: \( T = \begin{pmatrix} \cosh kL & \sinh kL \\ \frac{1}{b} \sinh kL & \frac{1}{b} \cosh kL \end{pmatrix} \)

- Both eigenvalues \( \lambda_1, \lambda_2 \) are real. Since \( \det T = 1/b \) and \( \text{Tr} T = (1 + \frac{1}{b}) \cosh kL > 2/\sqrt{b} \)

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- Since all transfer matrices are equal by construction, the \textbf{eigenvector} corresponding to \( \lambda_1 \) will give us an initial condition.

The eigenvalue \( \lambda_1 \) is given by

\[
\lambda_1 = \left( \frac{1}{2} + \frac{1}{2b} \right) \cosh kL - \sqrt{\left( \left( \frac{1}{2} + \frac{1}{2b} \right) \cosh kL \right)^2 - \frac{1}{b}} \\
= \frac{1}{\left( \frac{b}{2} + \frac{1}{2} \right) \cosh kL + \sqrt{\left( \left( \frac{b}{2} + \frac{1}{2} \right) \cosh kL \right)^2 - b}} \leq \frac{1}{b \cosh kL} \quad (3.2)
\]
the above solution is in $L^2$ for the tree since

$$\int_{\Gamma} |\phi|^2 = C \sum_n b^n \lambda_1^{2n}.$$ 

If $\lambda_1 < \alpha/\sqrt{b}$ for $\alpha < 1$ then the above sum converges.

The factor of $\frac{1}{b^n}$ makes the pointwise decay faster than the case of the line, where the decay is just $e^{-kx}$, but this comparison is “unfair.”
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If a solution is to be in $L^2$ for a tree the $1/\sqrt{b}$ factor is required for convergence. However, we have a factor of $1/b$ instead, which means that even if we consider partial integrals, the decay on the tree will be faster than on the line.
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Does this hold more generally?
The Agmon Metric for a General Tree

We label the maximum out-degree on the tree to be $d_{\text{max}}$ we will label each vertex by a $d_{\text{max}}$ary string $s$. We also denote the edge which terminates at vertex $v_s$ by $e_s$.

**Theorem 5**

Suppose $\psi \in L^2$ is a solution of $H$. Suppose at vertex $v_s$, $p_{sj}$ is the fraction of the derivative continuing down branch $e_{sj}$. For the path $P$ from the root to $x$, let

$$\rho(x) = \int_P (V(t) - E)_+ + 2 \sum_{v \in P} \delta_V(t) \log(1/p_v).$$

Then if $\psi \in L^2$, then $e^{(1-\epsilon)\rho} \psi$ is square integrable on the path $P$.

Note: $e^{(1-\epsilon)\rho} \psi$ is not in $L^2(\Gamma)$ but sufficient for pointwise bounds.
Key change

- For ingredient # 1 inequality $V - E - \left| \frac{F'}{F} \right|^2 > \delta$ has to be satisfied.
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- We sacrifice continuity.
- Get boundary terms in integration by parts. Recall that $\phi = F_\alpha \eta \psi$ so for an edge $e$ starting at $v_1$ and ending at $v_2$:

$$\int_e F\phi \frac{d^2}{dx^2} \left( \frac{1}{F} \phi \right) = \int_e \frac{d}{dx} \left( F^2 \eta \psi \frac{d}{dx} (\eta \psi) \right) - \int_e \frac{d}{dx} (F\phi) \frac{d}{dx} \left( \frac{1}{F} \phi \right)$$

$$= F^2 \eta \psi \frac{d}{dx} (\eta \psi) \bigg|_{v_1}^{v_2} - \int_e \frac{d}{dx} (F\phi) \frac{d}{dx} \left( \frac{1}{F} \phi \right)$$

- Want to take $F^2$ so that its discontinuity cancels with the discontinuity of $\frac{d}{dx} \psi$, so we can increase $F^2$ by a factor of $1/p$. 
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Technicality: we have to work with $F_\alpha$ so the discontinuity will be in $F_\alpha$ then take $\lim_{\alpha \to \infty}$
Case study: the harmonic millipede

We consider a millipede with segments of length $L$, at each vertex $v_k$ of which there dangles an infinitely long leg $e_k$. There is one leg at each vertex. The energy parameter is $E = -1$, no potential on the main path, so on each bodily edge, $\psi'' = \psi$. On the dangling legs the potential is $V(x) = (x + 2)^2 - 2$, where $x$ is the distance from $v_k$, and we see that $V - E > 0$ on the legs. The $L^2$ solutions on the legs solve 

$$\psi'' = (V(x) - (-1))\psi,$$

which easily gives $\text{cst.} e^{-(x+2)^2/2}$. We differentiate and find that $\psi'_{e_k}(v_k) = -2\psi(v_k)$. The transfer matrix for connecting solutions from one main-body segment to the next is

$$\begin{pmatrix}
\cosh L & \sinh L \\
\sinh L - 2 \cosh L & \cosh L - 2 \sinh L
\end{pmatrix}.$$

The eigenvalues of this are

$$\frac{e^L}{2} \left( 1 \pm \sqrt{1 - \left( \frac{2}{e^L} \right)^2} \right) \approx e^{\pm L}$$
Notice:

1. Not $L^2$ if any $p = 0$, so $1/p$’s are ok
2. This requires a lot of information on $\psi$

Consider instead an averaged function $\Psi$.

- Can use a similar Agmon argument to show that $F\Psi \in L^2$
- Do not need information on the $p$’s, but only get information
- Only get information on $\psi$ in an averaged sense
Consider a tree with equal edge lengths and equal branching numbers at each generation. Let $\psi \in L^2$ be an eigenfunction.
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Let $\Psi(x) = \sum_{y: \text{dist}(0, y) = x} w_y \psi(x)$

$w_y = \prod_v 1/b_v$ where the product is taken over all vertices $v$ on the path from 0 to $y$ and $b_v$ is the out-degree.

Notice: $\sum_{y: \text{dist}(0, y) = x} w_y = 1$, so indeed an average
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$\|\Psi(x)\|_{L^2}^2 \leq \int_0^\infty (\sum_y |w_y|^2)(\sum_y |\psi(y)|^2) \leq \|\psi(x)\|_{L^2}^2$

At each vertex can increase $F^2$ by a factor of $b_v$, so

$e(1-\epsilon) \int (V-E)^{1/2} + \frac{1}{2} \sum_n \log(b_n) \delta_v(t) dt \Psi$ is also in $L^2$. 
Questions to be settled:

1. Is the $L^2$ eigenfunction unique? Analogues of the the limit point-limit circle dichotomy.

2. What if we add leaves and finite subtrees to our regular infinite tree? “Obvious” that this will only improve the decay.

3. What happens in case of a more general graph? What if there are cycles?
Thank you!