An O(log m)-Competitive Algorithm for Online Machine Minimization

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Workshop on Approximation Algorithms and Parameterized Complexity
Banff, December 2015

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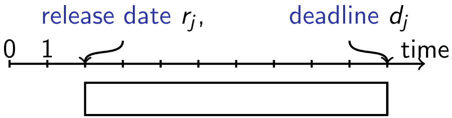

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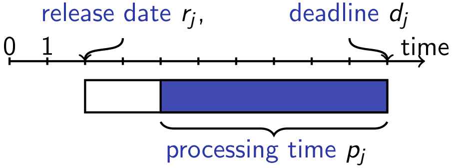

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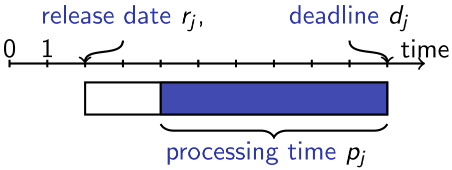

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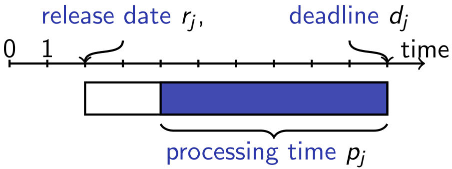

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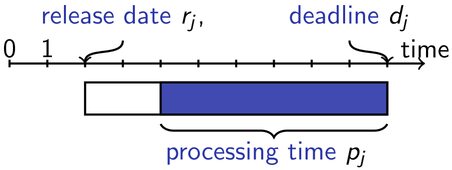

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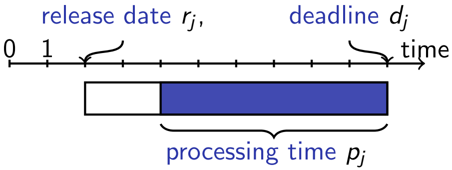

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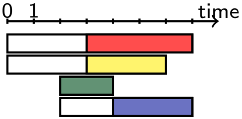

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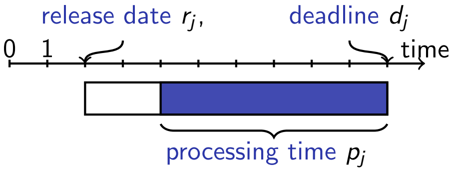

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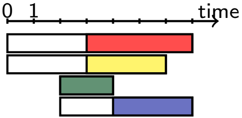

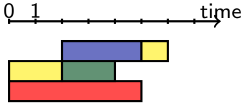

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Offline: Optimally solvable in polynomial time (LP or max flow). Horn '74

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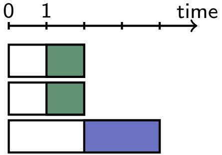

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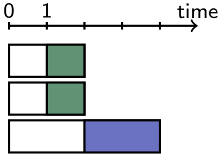

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The blue job is scheduled to some extent in $[0, 1]$, i.e., some green job unfinished by 1.

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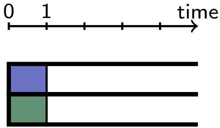

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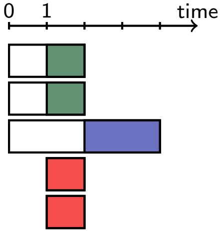

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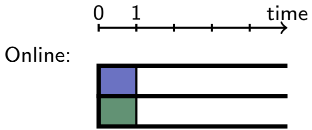

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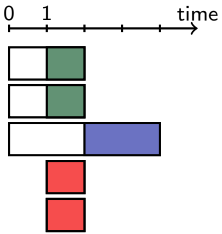

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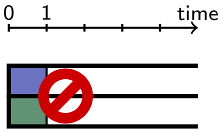

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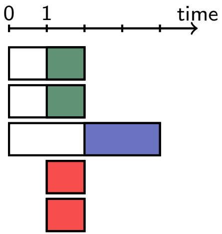

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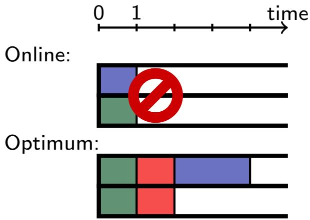

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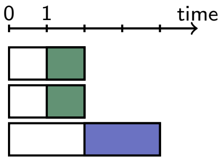

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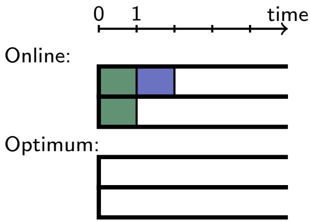

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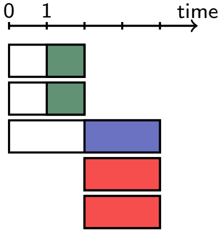

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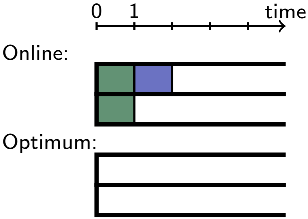

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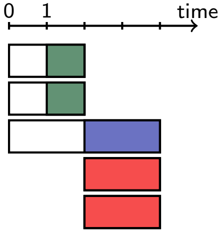

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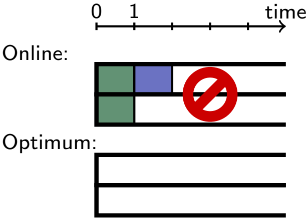

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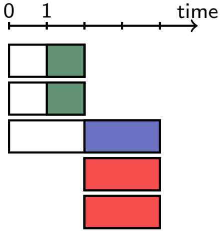

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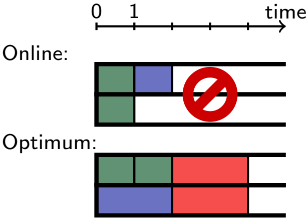

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Competitive analysis: An online algorithm ALG is c-competitive if

$$\mathrm{ALG}(I) \leq c \cdot m(I),$$

for all instances I with a feasible offline schedule on $m(I)$ machines.

Previous Results

Phillips et al. (STOC 1996)

- Best known algorithm (LLF) is $\mathcal{O}(\log \frac{p_{\max}}{p_{\min}})$-competitive.
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- Open if there is an $f(m)$-competitive algorithm for any fct. f.
- No better result even for $m = 2$.

Our Contribution

Theorem

There is an online algorithm with competitive ratio $\mathcal{O}(\log m)$.

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At the loss of a factor 4, we may assume m is known.

$\longrightarrow$ Guess-and-Double

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Algorithm EDF$_{m'}$: At any time schedule m' available jobs with minimum deadline and preempt other jobs if necessary.

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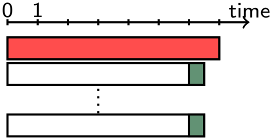

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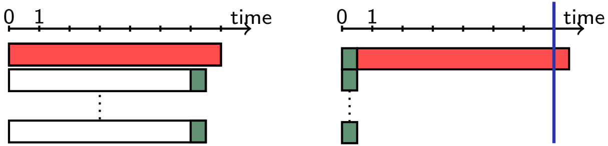

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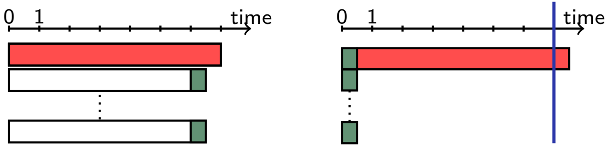

Def. Let $\alpha < 1$. Job j is α-tight if $p_j > \alpha(d_j - r_j)$ and α-loose othw.

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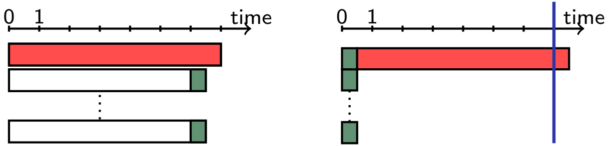

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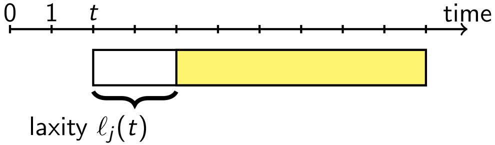
Theorem [Chen, M., Schewior (2015)]

If every job is α-loose, then $\mathrm{EDF}_\star$ is $\frac{1}{(1-\alpha)^2}$-competitive.

Least Laxity First (LLF)

Laxity of a job



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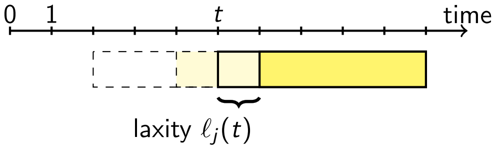

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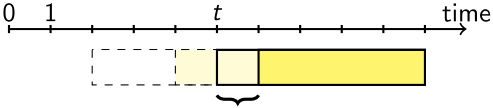

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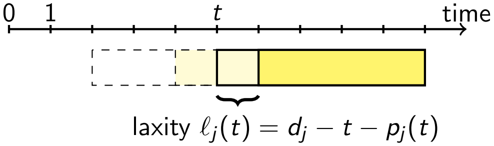

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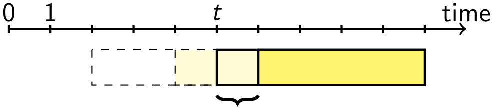

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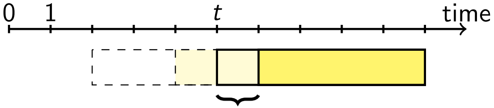

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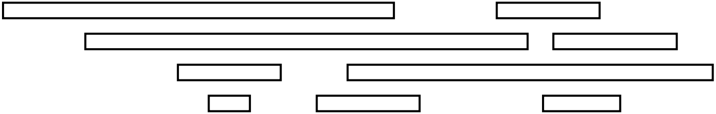
Theorem [Chen, M., Schewior 2015]

LLF may fail on $f(m)$ machines, for any f, even if jobs are α-tight.

A New Lower Bound – Tight Jobs Only

Load-based lower bounds.



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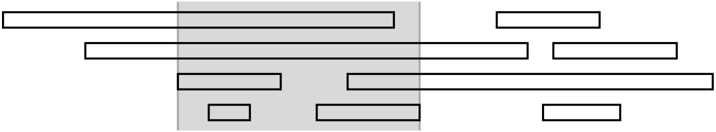

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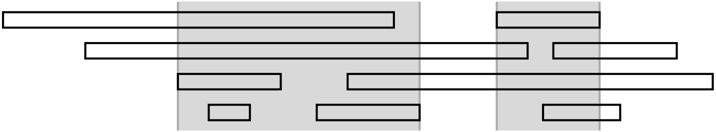

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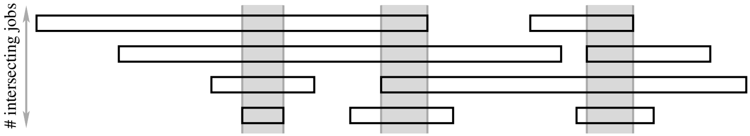
Relate laxity and number of intersecting intervals.



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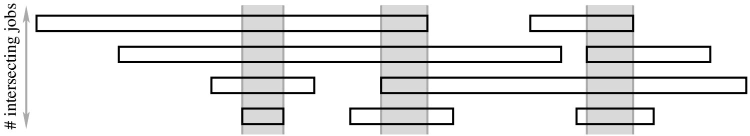

Definition: (μ, β)-critical pair (G, T)

(i) each time $t \in T$ is covered by $\geq \mu$ distinct jobs in G

(ii) $|T \cap I(j)| \geq \beta \cdot \ell_j$, for any $j \in G$

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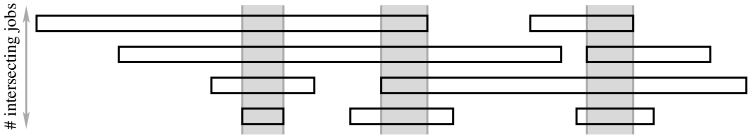

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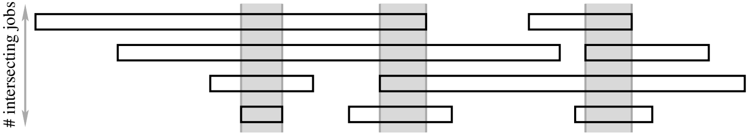

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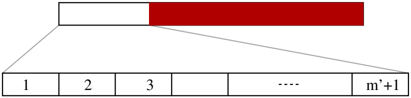
Given m: If there is a (μ, β)-critical pair then $\mu = \mathcal{O}(m \cdot \log \frac{1}{\beta})$.

Our Algorithm

Open m' machines. Charge the delay of a job to its laxity ($=$ budget).



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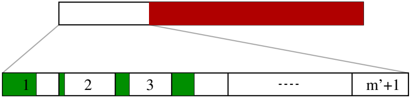

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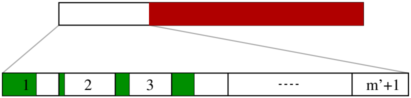

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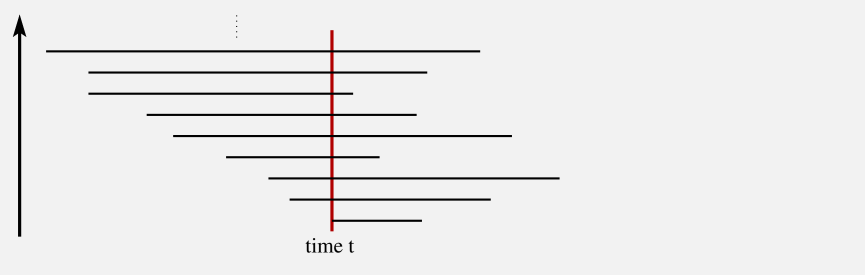

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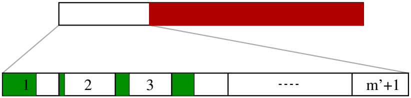

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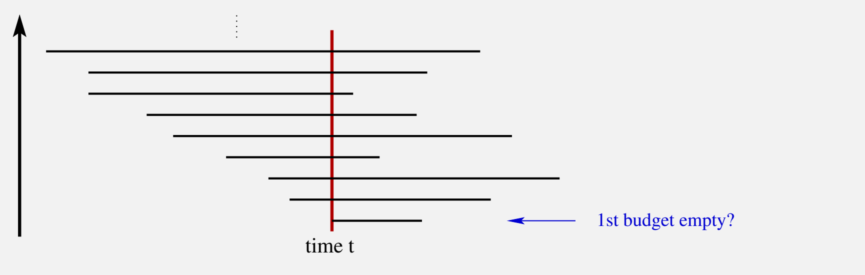

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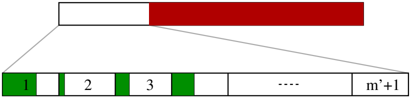

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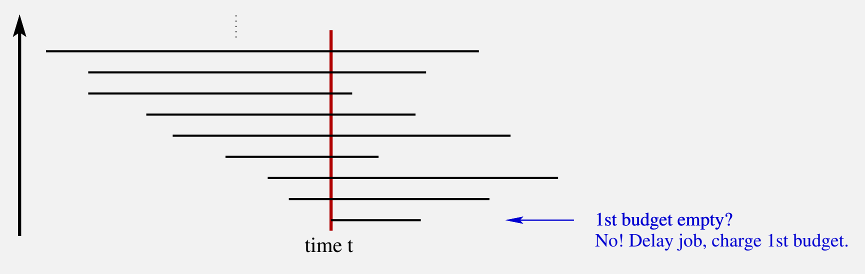

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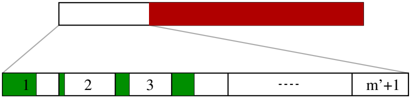

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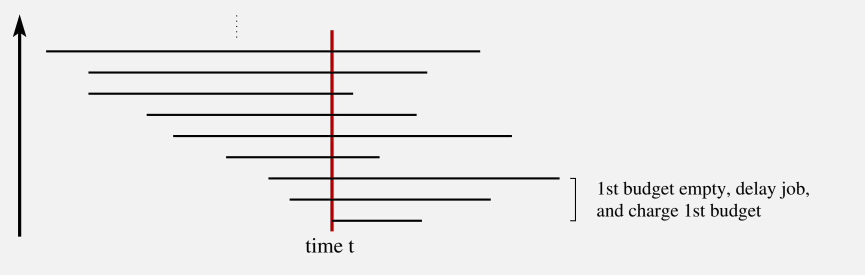

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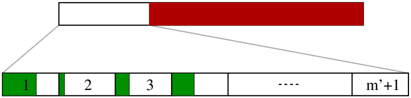

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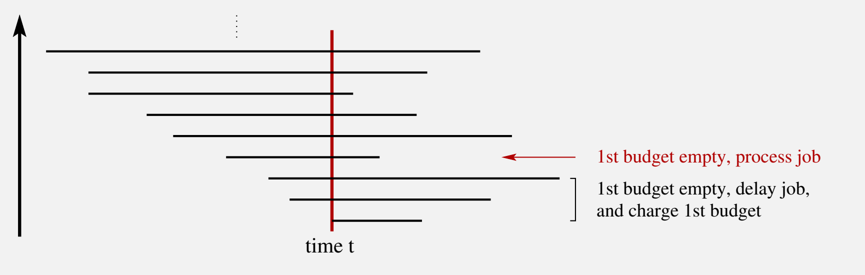

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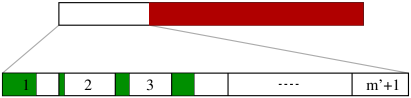

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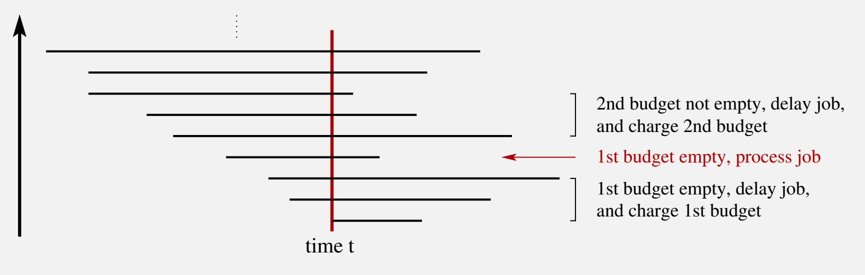

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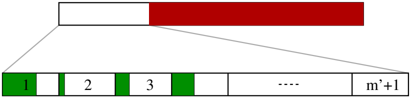

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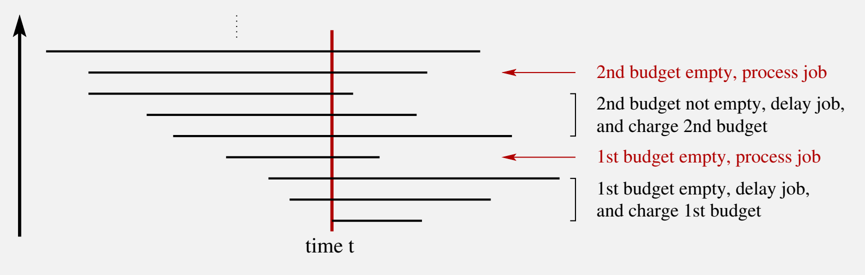

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Job set F

Time points
T

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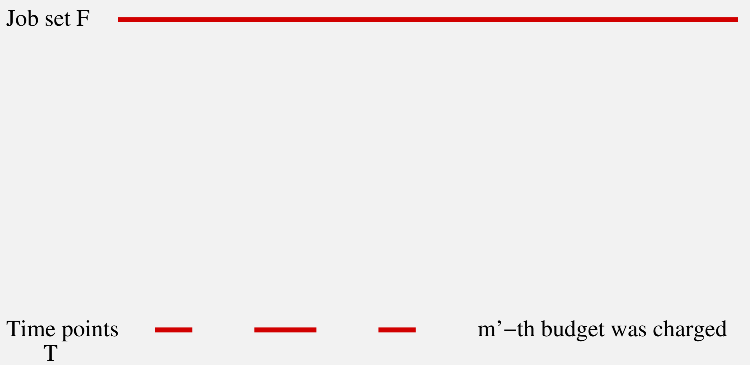

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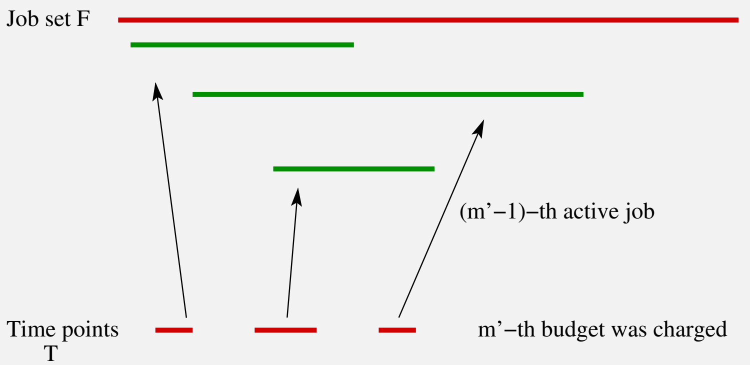

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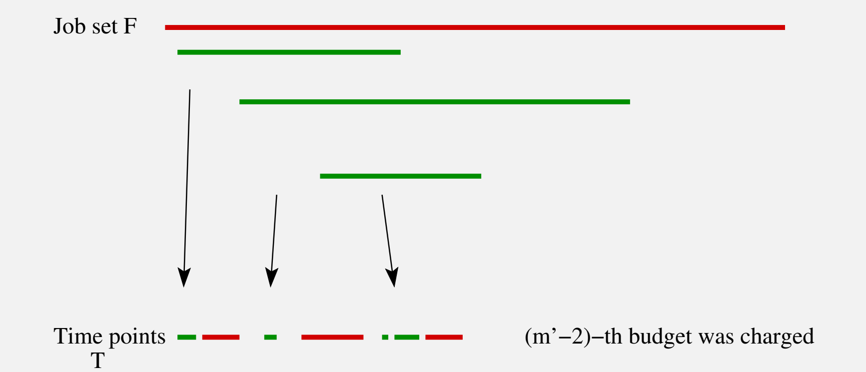

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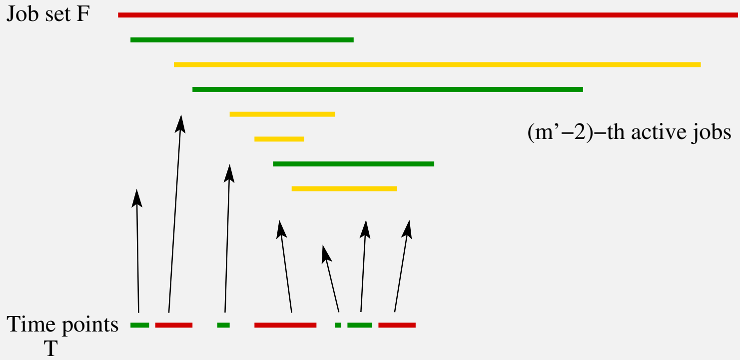

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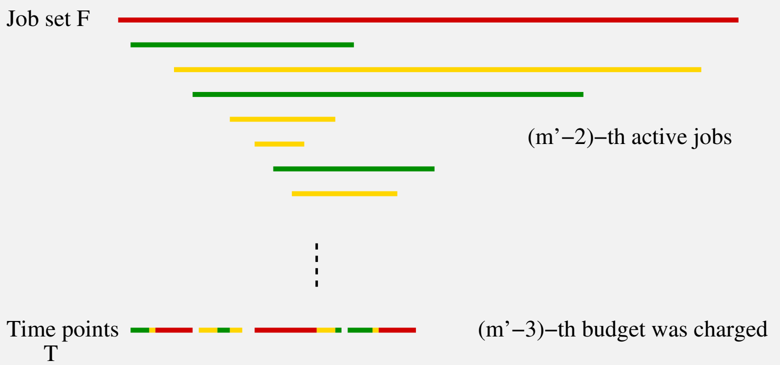

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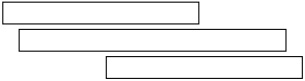
Theorem

Algorithm is $\mathcal{O}(\log m)$-competitive for online machine minimization.

Two major special cases

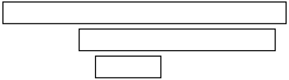
Agreeable instances

for any two jobs j and k: $r_j < r_k$ implies $d_j \leq d_k$



Laminar instances

for any two intersecting time windows I_j, I_k: either $I_j \subseteq I_k$ or $I_k \subseteq I_j$



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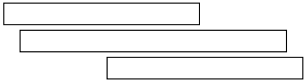

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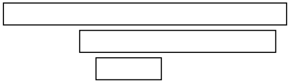

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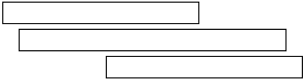

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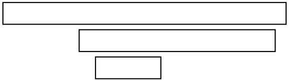

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$\longrightarrow$ slightly modified lower bound and failure-set construction

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