Chain rules for quantum Rényi entropies

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Outline

- Introduction
- Quantum Rényi entropies: background and properties
- The Hadamard three-line theorem
- The proof

Rényi entropy

Rényi entropy of order α:

$$H_\alpha(X) = \frac{1}{1-\alpha} \log \left(\sum_x p_X(x)^\alpha \right)$$

Quantum version:

$$H_\alpha(A)_\rho = \frac{1}{1-\alpha} \log \mathrm{Tr}[\rho_A^\alpha]$$

Rényi divergence

$$D_\alpha(p\|q) = \frac{1}{\alpha-1}\log\left(\sum_x p(x)^\alpha q(x)^{1-\alpha}\right)$$

We can use this to define a conditional Rényi entropy:

$$\begin{aligned} H_\alpha(X|Y)_p &= -\inf_{q_Y} D_\alpha(p_{XY}\|\mathbb{1}_X \otimes q_Y) \\ &= \sup_{q_Y} \frac{1}{1-\alpha}\log\left(\sum_{xy} p_{XY}(x,y)^\alpha q_Y(y)^\alpha\right). \end{aligned}$$

How do we get a quantum version?

Rényi divergence

If ρ and σ don't commute, many possible definitions:

$$
\begin{aligned}
D_\alpha(\rho\|\sigma) &= \frac{1}{\alpha-1}\log\mathrm{Tr}\left[\rho^{\alpha}\sigma^{1-\alpha}\right] \\
D_\alpha(\rho\|\sigma) &= \frac{1}{\alpha-1}\log\mathrm{Tr}\left[\rho^{\frac{\alpha}{2}}\sigma^{\frac{1-\alpha}{2}}\rho^{\frac{\alpha}{2}}\sigma^{\frac{1-\alpha}{2}}\right] \\
D_\alpha(\rho\|\sigma) &= \frac{1}{\alpha-1}\log\mathrm{Tr}\left[\left(\sigma^{\frac{1-\alpha}{2\alpha}}\rho\sigma^{\frac{1-\alpha}{2\alpha}}\right)^{\alpha}\right] \\
&\vdots
\end{aligned}
$$

How to pick the right one?

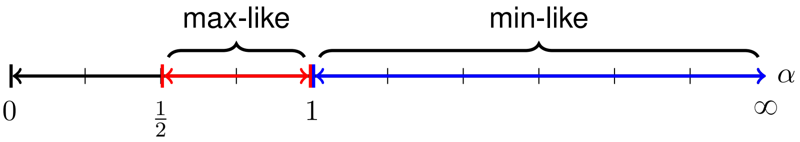
“Sandwiched” Rényi entropy

$$H_\alpha(A|B)_\rho \quad = \quad -\inf_{\sigma_B} D_\alpha(\rho_{AB} \| \mathbb{1}_A \otimes \sigma_B)$$

Properties:

- $\lim_{\alpha\to\infty} H_\alpha(A|B)_\rho = H_{\min}(A|B)_\rho$
- $\lim_{\alpha\to 1} H_\alpha(A|B)_\rho = H(A|B)_\rho$
- $H_{\frac{1}{2}}(A|B)_\rho = H_{\max}(A|B)_\rho$
- Monotone decreasing in α: if $\frac{1}{2} \leqslant \alpha \leqslant \gamma$, then $H_\alpha(A|B)_\rho \geqslant H_\gamma(A|B)_\rho$
- Data processing: $H_\alpha(A|BC)_\rho \leqslant H_\alpha(A|B)_\rho$ for $\alpha \geqslant \frac{1}{2}$.
- Duality: for pure ψ_{ABC}, $H_\alpha(A|B)_\psi = -H_\gamma(A|C)_\psi$, where $\frac{1}{\alpha} + \frac{1}{\gamma} = 2$.

Sandwiched Rényi entropy: range of α



- $\alpha < 1$: "max-like" entropies: sensitive to large numbers of small eigenvalues
- $\alpha > 1$: "min-like" entropies: sensitive to small numbers of large eigenvalues
- Duality: for pure ψ_{ABC}, $H_\alpha(A|B)_\psi = -H_\gamma(A|C)_\psi$, with $\frac{1}{\alpha} + \frac{1}{\gamma} = 2$.

Sandwiched Rényi entropy: applications

- Strong converses:
  - M. M. Wilde, A. Winter, and D. Yang, arXiv: `1306.1586`
  - T. Cooney, M. Mosonyi, and M. M. Wilde, arXiv: `1408.3373`
  - M. Tomamichel, M. M. Wilde, and A. Winter, arXiv: `1406.2946`
- Hypothesis testing
  - M. Mosonyi and T. Ogawa, arXiv: `1309.3228`
  - M. Mosonyi and T. Ogawa, arXiv: `1409.3562`
- ...

Chain rules

Goal of this talk: prove chain rules for sandwiched Rényi entropies.
Given a state ρ_{ABC} and α, β, γ with $\frac{\alpha}{\alpha-1} = \frac{\beta}{\beta-1} + \frac{\gamma}{\gamma-1}$:

$$H_\alpha(AB|C)_\rho \geqslant H_\beta(A|BC)_\rho + H_\gamma(B|C)_\rho$$

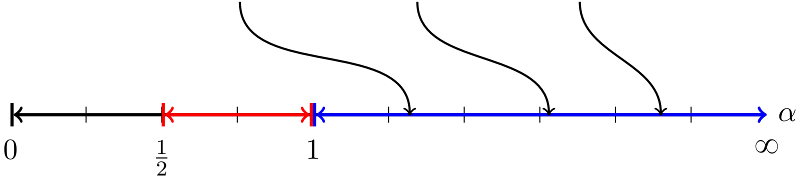
if $\alpha, \beta, \gamma > 1$, and

$$H_\alpha(AB|C)_\rho \leqslant H_\beta(A|BC)_\rho + H_\gamma(B|C)_\rho$$

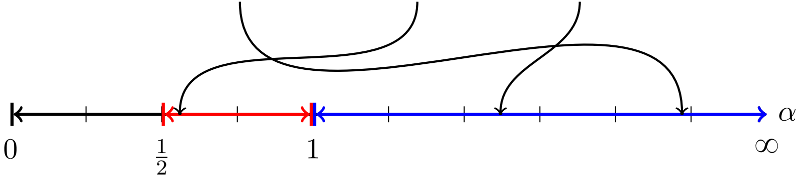
if $\alpha > 1$, and one of β, γ is < 1.

Chain rules: parameters

$$H_\alpha(AB|C) \geqslant H_\beta(A|BC) + H_\gamma(B|C)$$

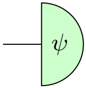


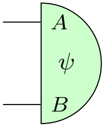
$$H_\alpha(AB|C) \leqslant H_\beta(A|BC) + H_\gamma(B|C)$$

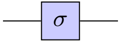


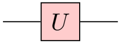
The proof

- Graphical calculus and expressions for H_α (+bonus proof of duality)
- Hadamard three-line theorem (+bonus proof of monotonicity in α)
- Actual proof

$|\psi\rangle =$ 

$|\psi\rangle_{AB} =$ 

$\sigma =$ 

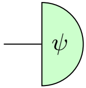
$U =$ 

Graphical calculus

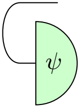
Turning kets into bras:

$$|\psi\rangle = \sum_i a_i |i\rangle$$

$$=$$



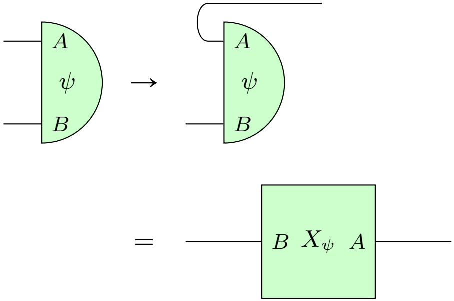
$$\rightarrow$$



$$= \sum_i a_i \langle i|.$$

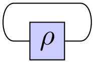
Graphical calculus

$$|\psi\rangle_{AB} = \sum_{ij} \lambda_{ij} |i\rangle_A \otimes |j\rangle_B \quad \rightarrow \quad X_\psi = \sum_{ij} \lambda_{ij} |j\rangle_B \langle i|_A$$

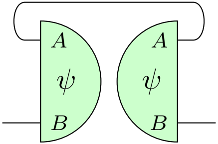


Graphical calculus: trace

$\mathrm{Tr}[\rho] =$



$\mathrm{Tr}_A[|\psi\rangle\langle\psi|_{AB}] =$



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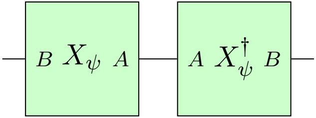


Schatten p-norms

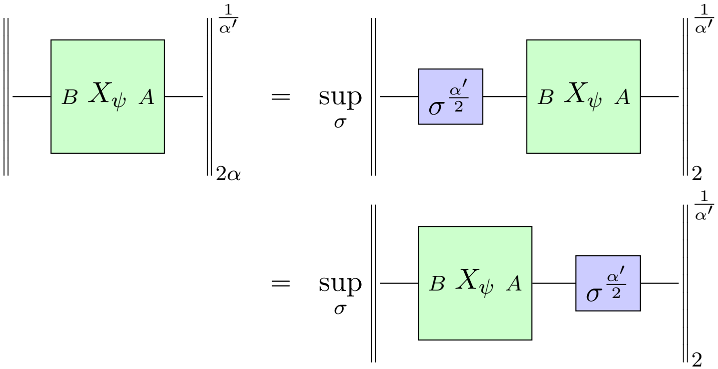
Definition:

$$\begin{aligned}\|X\|_p &:= \operatorname{Tr}\left[|X|^p\right]^{1/p} \\ &= \operatorname{Tr}\left[(XX^\dagger)^{p/2}\right]^{1/p}\end{aligned}$$

Dual norm: let $\frac{1}{p} + \frac{1}{q} = 1$.

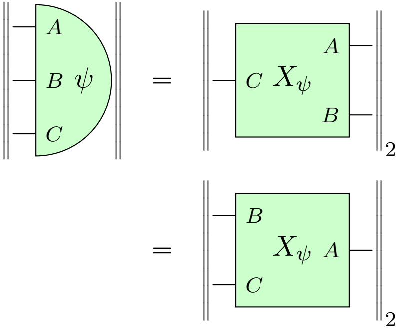
$$\begin{aligned}\|X_{A\to B}\|_p &= \sup_{Y_{A\to B}:\|Y\|_q\leqslant 1} \operatorname{Tr}\left[Y^\dagger X\right] \\ &= \sup_{\sigma\geqslant 0:\operatorname{Tr}[\sigma]\leqslant 1} \operatorname{Tr}\left[\sigma\right]\end{aligned}$$

Lemma: If $\alpha' = 1 - \frac{1}{\alpha}$, we have that:



We will be doing lots of this.

Another useful fact:

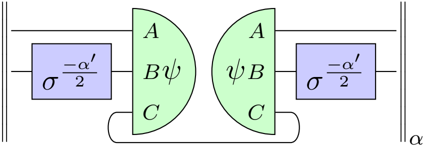


Let ψ_{ABC} be pure. Then,

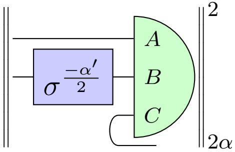
$$H_\alpha(A|B)_\psi = \sup_\sigma \frac{\alpha}{1-\alpha} \log \mathrm{Tr}\left[\left(\sigma_B^{\frac{-\alpha'}{2}} \psi_{AB} \sigma_B^{\frac{-\alpha'}{2}}\right)^\alpha\right]^{\frac{1}{\alpha}}$$

$$= \sup_\sigma \frac{-1}{\alpha'} \log \left\| \sigma_B^{\frac{-\alpha'}{2}} \psi_{AB} \sigma_B^{\frac{-\alpha'}{2}} \right\|_\alpha$$

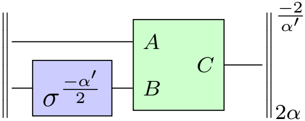
$$= \sup_\sigma \frac{-1}{\alpha'} \log$$

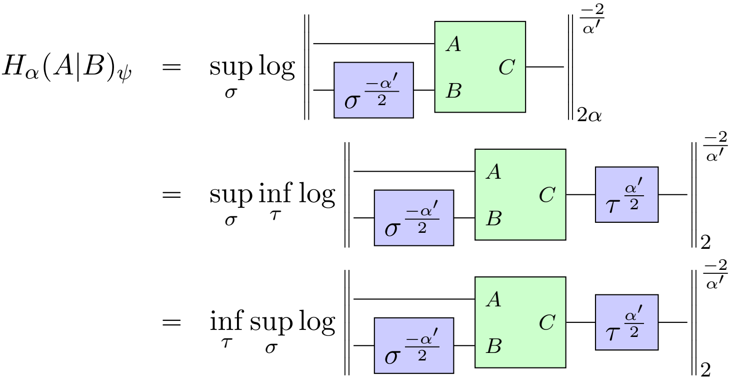


$$= \sup_\sigma \frac{-1}{\alpha'} \log$$



$$= \sup_\sigma \log$$



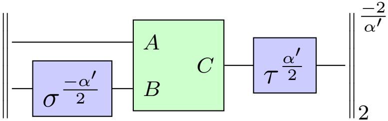


$H_\alpha(A|B)_\psi = \sup_\sigma \log$
$\sigma^{\frac{-\alpha'}{2}}$
A
B
C
2α
$\frac{-2}{\alpha'}$
$= \sup_\sigma \inf_\tau \log$
$\tau^{\frac{\alpha'}{2}}$
2
$= \inf_\tau \sup_\sigma \log$

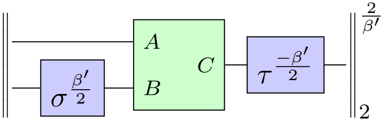
First benefit: duality

Let α and β be such that $\frac{1}{\alpha} + \frac{1}{\beta} = 2$. Then, $\beta' = -\alpha'$. We have

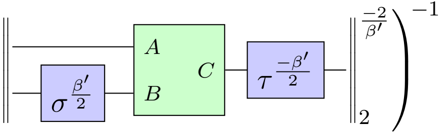
$H_\alpha(A|B)_\psi = \log \sup_\sigma \inf_\tau$



$= \log \sup_\sigma \inf_\tau$



$= \log \left(\inf_\sigma \sup_\tau \right.$

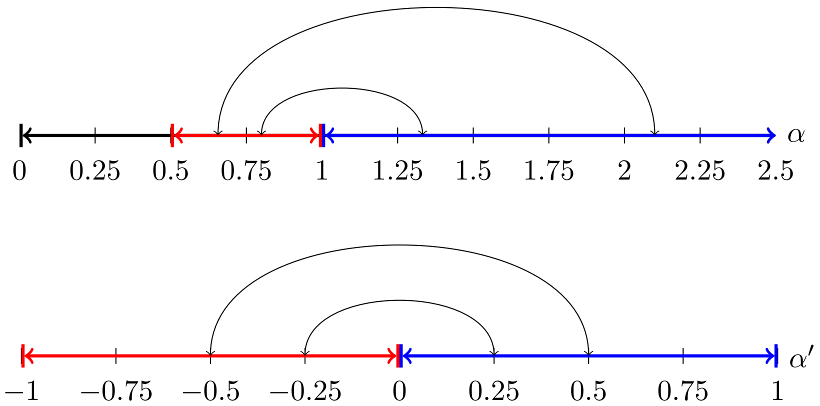


$= -H_\beta(A|C)_\psi$

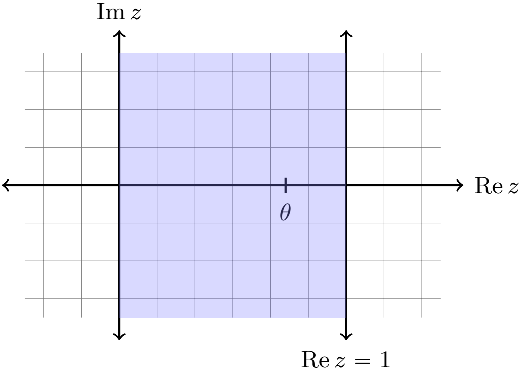
Duality

$$H_\alpha(A|B)_\psi = -H_\beta(A|C)_\psi,$$

with $\frac{1}{\alpha} + \frac{1}{\beta} = 2 \Rightarrow \beta' = -\alpha'$.

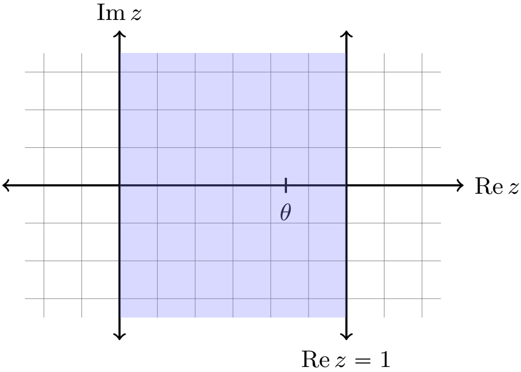


Hadamard's three-line theorem



- $F(z)$: holomorphic on interior of blue area, continuous at boundary
- Theorem: $|F(\theta)| \leqslant \left(\sup_{t\in\mathbb{R}} |F(it)|\right)^{1-\theta} \left(\sup_{t\in\mathbb{R}} |F(1+it)|\right)^{\theta}$
- Proof: maximum modulus principle

Hadamard's three-line theorem



- $F(z) : \mathbb{C} \to \mathbb{C}^{m \times n}$: holomorphic on interior, continuous
- Theorem: $\|F(\theta)\|_{p_\theta} \leqslant (\sup_t \|F(it)\|_{p_0})^{1-\theta} (\sup_t \|F(1+it)\|_{p_1})^{\theta}$
- $\frac{1}{p_\theta} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$

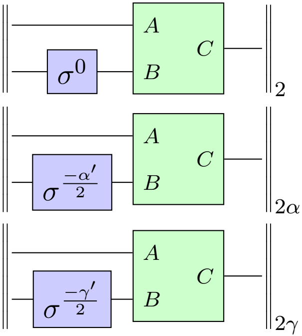
Monotonicity in α

Let $1 < \alpha < \gamma$. We want to prove that $H_\alpha(A|B)_\psi \geqslant H_\gamma(A|B)_\psi$.

$$1 =$$

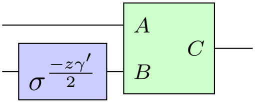
$$2^{\frac{-\alpha'}{2} H_\alpha(A|B)_{\rho|\sigma}} =$$

$$2^{\frac{-\gamma'}{2} H_\gamma(A|B)_{\rho|\sigma}} =$$



Choose: $F(z) = \sigma^{\frac{-z\gamma'}{2}} X \qquad p_0 = 2 \qquad p_1 = 2\gamma \qquad \theta = \frac{\alpha'}{\gamma'}$

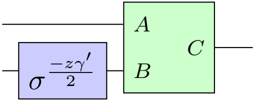
$F(z) =$



$$
\begin{aligned}
p_0 &= 2 \\
p_1 &= 2\gamma \\
\theta &= \frac{\alpha'}{\gamma'}
\end{aligned}
$$

We can calculate that $p_\theta = 2\alpha$, and $\|F(\theta)\|_{2\alpha} = 2^{\frac{-\alpha'}{2} H_\alpha(A|B)_{\rho|\sigma}}$.

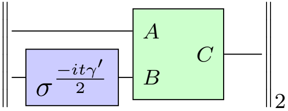
$F(z) =$

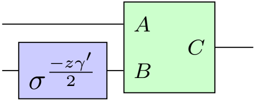


From the three-line theorem:

$$\|F(\theta)\|_{2\alpha} \leqslant \left(\sup_{t\in\mathbb{R}} \|F(it)\|_2 \right)^{1-\frac{\alpha'}{\gamma'}} \left(\sup_{t\in\mathbb{R}} \|F(1+it)\|_{2\gamma} \right)^{\frac{\alpha'}{\gamma'}}$$

$\|F(it)\|_2 =$



$F(z) =$ 

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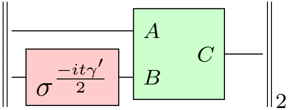
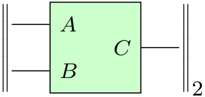
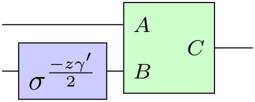
$\|F(it)\|_2 =$ 

$=$ 

$= 1$

$$F(z) \quad =$$



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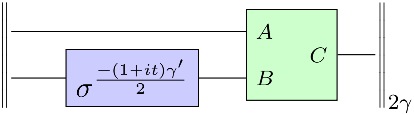
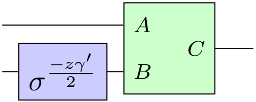
$$\|F(\theta)\|_{2\alpha} \leqslant \left(\sup_{t\in\mathbb{R}} \|F(it)\|_2\right)^{1-\frac{\alpha'}{\gamma'}} \left(\sup_{t\in\mathbb{R}} \|F(1+it)\|_{2\gamma}\right)^{\frac{\alpha'}{\gamma'}}$$

$$\|F(1+it)\|_{2\gamma} \quad =$$



$F(z) =$



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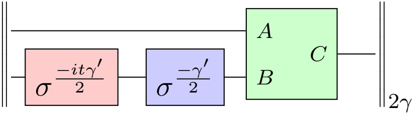
$\|F(1+it)\|_{2\gamma} =$



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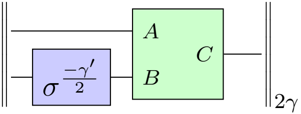


$= 2^{\frac{-\gamma'}{2} H_\gamma(A|B)_{\rho|\sigma}}$

Putting it all together:

$$\begin{aligned} 2^{\frac{-\alpha'}{2} H_\alpha(A|B)_{\rho|\sigma}} &= \|F(\theta)\|_{2\alpha} \\ &\leqslant \left(2^{\frac{-\gamma'}{2} H_\gamma(A|B)_{\rho|\sigma}}\right)^{\frac{\alpha'}{\gamma'}} \\ &= 2^{\frac{-\alpha'}{2} H_\gamma(A|B)_{\rho|\sigma}} \end{aligned}$$

and therefore

$$\begin{aligned} H_\alpha(A|B)_{\rho|\sigma} &\geqslant H_\gamma(A|B)_{\rho|\sigma} \\ H_\alpha(A|B)_{\rho} &\geqslant H_\gamma(A|B)_{\rho}. \end{aligned}$$

We can use slightly different parameter choices to show this for $\frac{1}{2} \leqslant \alpha \leqslant \gamma$.

Chain rules

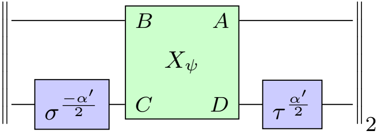
And now for the main event: we will prove that

$$H_\alpha(AB|C) \geqslant H_\beta(A|BC) + H_\gamma(B|C)$$

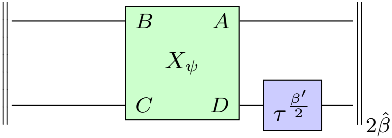
with $\alpha, \beta, \gamma > 1$ and

$$\frac{1}{\alpha'} = \frac{1}{\beta'} + \frac{1}{\gamma'}$$

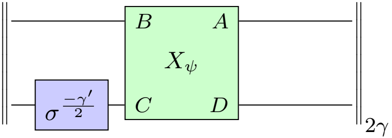
$$2^{\frac{-\alpha'}{2} H_\alpha(AB|C)_{\rho|\sigma}} = \sup_\tau$$



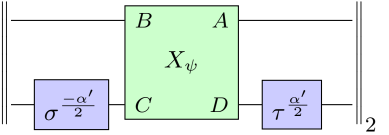
$$2^{\frac{-\beta'}{2} H_\beta(A|BC)_{\rho}} = \sup_\tau$$



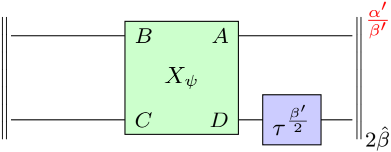
$$2^{\frac{-\gamma'}{2} H_\gamma(B|C)_{\rho|\sigma}} =$$



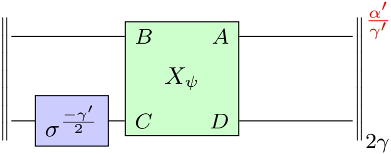
$$2^{\frac{-\alpha'}{2} H_\alpha(AB|C)_{\rho|\sigma}} = \sup_\tau$$



$$2^{\frac{-\alpha'}{2} H_\beta(A|BC)_{\rho}} = \sup_\tau$$



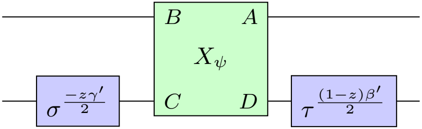
$$2^{\frac{-\alpha'}{2} H_\gamma(B|C)_{\rho|\sigma}} =$$



Choose

$$F(z) = \sigma^{\frac{-z\gamma'}{2}} X_\psi \tau^{\frac{(1-z)\beta'}{2}} \qquad \theta = \frac{\alpha'}{\gamma'} \qquad 1 - \theta = \frac{\alpha'}{\beta'}$$

$F(z) =$

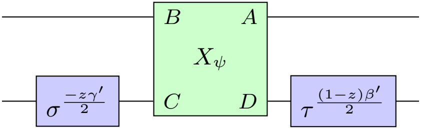


$$
\begin{aligned}
\theta &= \frac{\alpha'}{\gamma'} \\
1-\theta &= \frac{\alpha'}{\beta'} \\
p_0 &= 2\hat{\beta} \\
p_1 &= 2\gamma
\end{aligned}
$$

This is true if $\frac{\alpha'}{\beta'} + \frac{\alpha'}{\gamma'} = 1$, so $\frac{1}{\alpha'} = \frac{1}{\beta'} + \frac{1}{\gamma'}$.

Recall:

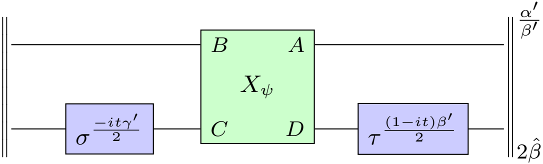
$F(z) \quad =$



From the 3-line theorem:

$$\|F(\theta)\|_{p_\theta} \leqslant \left(\sup_t \|F(it)\|_{p_0}\right)^{1-\theta} \left(\sup_t \|F(1+it)\|_{p_1}\right)^{\theta}$$

$\|F(it)\|_{p_0}^{1-\theta} \quad =$



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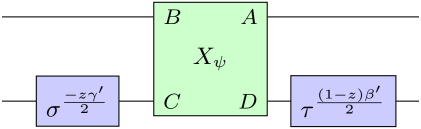

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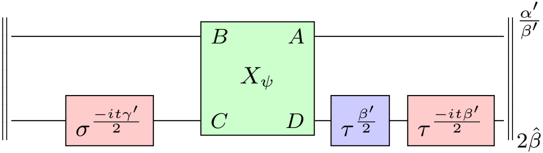
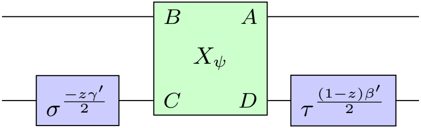


Recall:

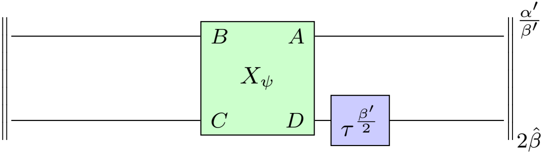
$F(z) =$



From the 3-line theorem:

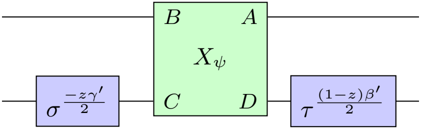
$$\|F(\theta)\|_{p_\theta} \leqslant \left(\sup_t \|F(it)\|_{p_0}\right)^{1-\theta} \left(\sup_t \|F(1+it)\|_{p_1}\right)^{\theta}$$

$\|F(it)\|_{p_0}^{1-\theta} =$



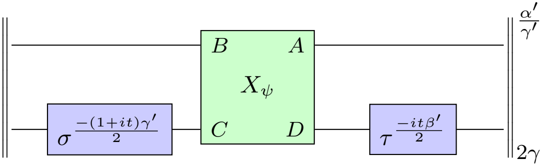
$$= \|F(0)\|_{p_0}^{1-\theta} = 2^{\frac{-\alpha'}{2} H_\beta(A|BC)_\rho}$$

Recall:

$F(z) =$ 

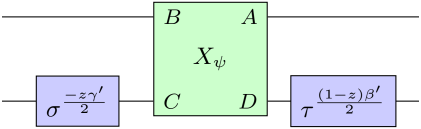
From the 3-line theorem:

$$\|F(\theta)\|_{p_\theta} \leqslant \left(\sup_t \|F(it)\|_{p_0}\right)^{1-\theta} \left(\sup_t \|F(1+it)\|_{p_1}\right)^{\theta}$$

$\|F(1+it)\|_{p_1}^{\theta} =$ 

Recall:

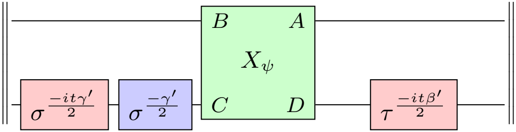
$$F(z) =$$



From the 3-line theorem:

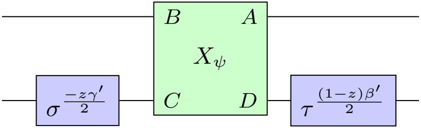
$$\|F(\theta)\|_{p_\theta} \leqslant \left(\sup_t \|F(it)\|_{p_0}\right)^{1-\theta} \left(\sup_t \|F(1+it)\|_{p_1}\right)^{\theta}$$

$$\|F(1+it)\|_{p_1}^{\theta} = \left\| \quad \right\|_{2\gamma}^{\frac{\alpha'}{\gamma'}}$$



Recall:

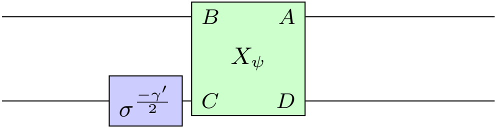
$$F(z) \quad = $$



From the 3-line theorem:

$$\|F(\theta)\|_{p_\theta} \quad \leqslant \quad \left(\sup_t \|F(it)\|_{p_0}\right)^{1-\theta} \left(\sup_t \|F(1+it)\|_{p_1}\right)^{\theta}$$

$$\|F(1+it)\|_{p_1}^{\theta} \quad = \quad \left\| \quad \right\|_{2\gamma}^{\frac{\alpha'}{\gamma'}}$$



$$= \quad \|F(1)\|_{p_1}^{\theta} = 2^{\frac{-\alpha'}{2} H_\gamma(B|C)_{\rho|\sigma}}$$

So we've proven that

$$H_\alpha(AB|C) \geqslant H_\beta(A|BC) + H_\gamma(B|C)$$

for $\alpha, \beta, \gamma > 1$, where

$$\frac{1}{\alpha'} = \frac{1}{\beta'} + \frac{1}{\gamma'}.$$

By duality, if $\alpha, \beta, \gamma < 1$, we have that

$$H_\alpha(AB|C) \leqslant H_\beta(A|BC) + H_\gamma(B|C).$$

How about the other cases?

Chain rules: the second case

Now, suppose that $\alpha > 1$, and one of β, γ is below 1. We will show that

$$H_\alpha(AB|C) \leqslant H_\beta(A|BC) + H_\gamma(B|C),$$

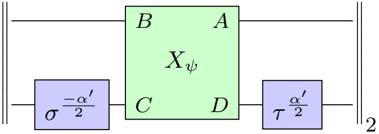
with

$$\frac{1}{\alpha'} = \frac{1}{\beta'} + \frac{1}{\gamma'}$$

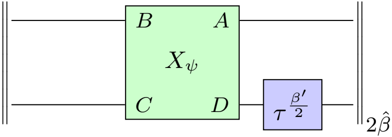
as before.

Goal: $H_\beta(A|BC) \geqslant H_\alpha(AB|C) - H_\gamma(B|C)$

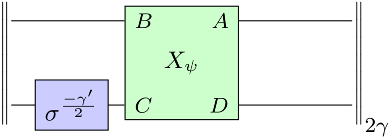
$$2^{\frac{-\alpha'}{2} H_\alpha(AB|C)_{\rho|\sigma}} = \sup_{\tau}$$



$$2^{\frac{-\beta'}{2} H_\beta(A|BC)_{\rho}} = \sup_{\tau}$$

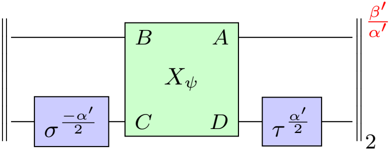


$$2^{\frac{-\gamma'}{2} H_\gamma(B|C)_{\rho|\sigma}} =$$

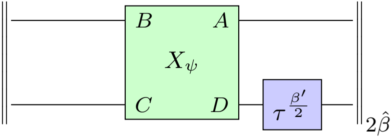


Goal: $H_\beta(A|BC) \geqslant H_\alpha(AB|C) - H_\gamma(B|C)$

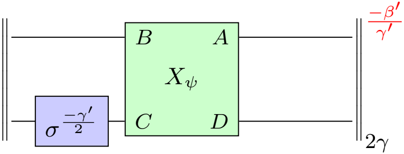
$2^{\frac{-\beta'}{2} H_\alpha(AB|C)_{\rho|\sigma}} = \sup_\tau$



$2^{\frac{-\beta'}{2} H_\beta(A|BC)_\rho} = \sup_\tau$



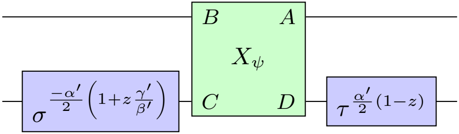
$2^{\frac{\beta'}{2} H_\gamma(B|C)_{\rho|\sigma}} =$



Choose

$F(z) = \sigma^{\frac{-\alpha'}{2}\left(1+\frac{z\gamma'}{\beta'}\right)} X_\psi \tau^{\frac{\alpha'}{2}(1-z)} \qquad \theta = \frac{-\beta'}{\gamma'} \qquad 1-\theta = \frac{\beta'}{\alpha'}$

$F(z) =$



$$\theta = \frac{-\beta'}{\gamma'}$$

$$1-\theta = \frac{\beta'}{\alpha'}$$

$$p_0 = 2$$

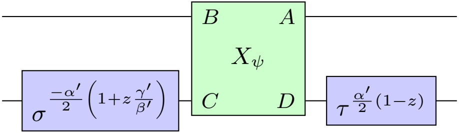
$$p_\theta = 2\hat{\beta}$$

$$p_1 = 2\gamma$$

This is true if $\frac{\alpha'}{\beta'} + \frac{\alpha'}{\gamma'} = 1$, so $\frac{1}{\alpha'} = \frac{1}{\beta'} + \frac{1}{\gamma'}$.

Recall:

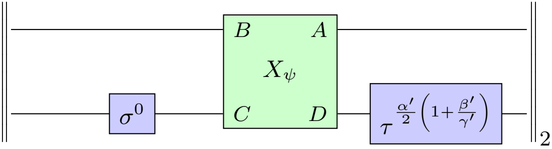
$$F(z) =$$



From the 3-line theorem:

$$\|F(\theta)\|_{p_\theta} \leqslant \left(\sup_t \|F(it)\|_{p_0}\right)^{1-\theta} \left(\sup_t \|F(1+it)\|_{p_1}\right)^{\theta}$$

$$\|F(\theta)\|_{p_\theta} =$$



Recall:

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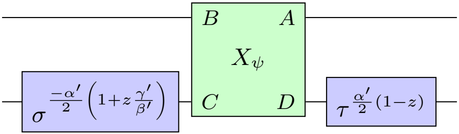

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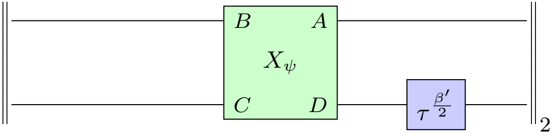
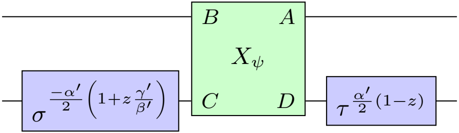


$$= 2^{\frac{-\beta'}{2} H_\beta(A|BC)_\rho}$$

Recall:

$$F(z) \;=$$



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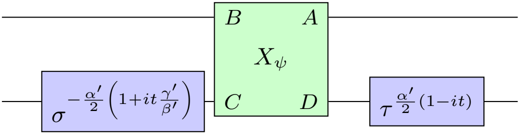
$$\|F(it)\|_{p_0}^{1-\theta} = \left\| \quad \right\|_2^{\frac{\beta'}{\alpha'}}$$



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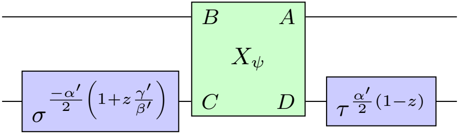
$$F(z) \quad =$$



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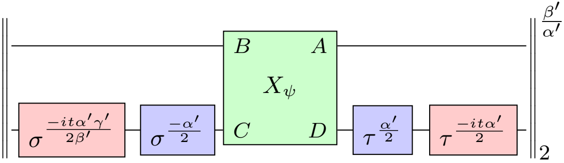

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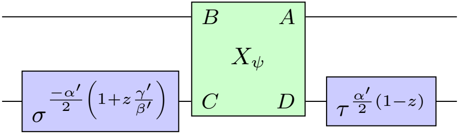

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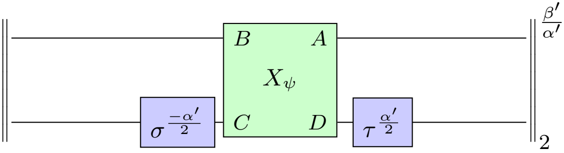

$$= \quad \|F(0)\|_{p_0}^{1-\theta} = 2^{\frac{-\beta'}{2} H_\alpha(AB|C)_{\rho|\sigma}}$$

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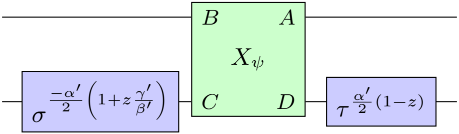

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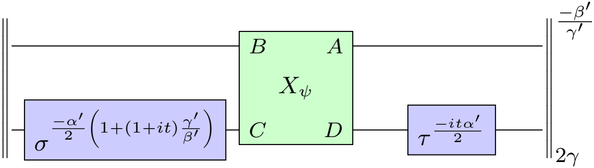

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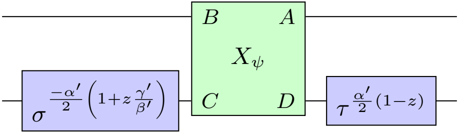
$F(z) \quad =$ 

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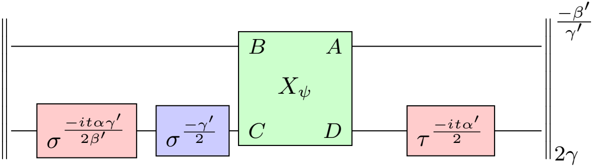
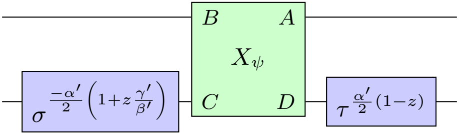
$\|F(1+it)\|_{p_1}^{\theta} \quad =$ 

Recall:

$F(z) \;=$



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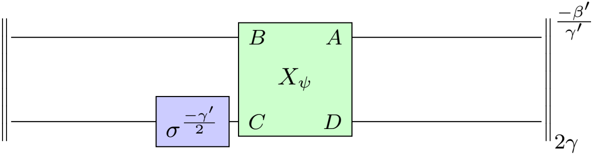
$\|F(1+it)\|_{p_1}^{\theta} \;=$



$$= \|F(1)\|_{p_1}^{\theta} = 2^{\frac{\beta'}{2}H_\gamma(B|C)_{\rho|\sigma}}$$

So we've proven that

$$H_\beta(A|BC) \geqslant H_\alpha(AB|C) - H_\gamma(B|C),$$

or, to reformulate,

$$H_\alpha(AB|C) \leqslant H_\beta(A|BC) + H_\gamma(B|C)$$

for $\alpha > 1$, β' and γ' having opposite signs, and where

$$\frac{1}{\alpha'} = \frac{1}{\beta'} + \frac{1}{\gamma'}.$$

By duality, if $\alpha < 1$:

$$H_\alpha(AB|C) \geqslant H_\beta(A|BC) + H_\gamma(B|C).$$

Further work and open questions

In the same vein:

- Bounds on recoverability using the three-line theorem (Mark's talk)
- "Swiveled" Rényi entropies (arXiv: `1506.00981`)

Open questions:

- Chain rules for (conditional) Rényi mutual information?

The papers

Main result from

- F. Dupuis, “Chain rules for quantum Rényi entropies”, arXiv: `1410.5455`

with material from

- S. Beigi, “Sandwiched Rényi divergence satisfies data processing inequality”, arXiv: `1306.5920`
- M. Müller-Lennert, F. Dupuis, O. Szehr, S. Fehr, and M. Tomamichel, “On quantum Rényi entropies: a new generalization and some properties”, arXiv: `1306.3142`
- M. M. Wilde, A. Winter, and D. Yang, “Strong converse for the classical capacity of entanglement-breaking and Hadamard channels via a sandwiched Rényi relative entropy”, arXiv: `1306.1586`

The end