

# Chain rules for quantum Rényi entropies

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# Outline

- Introduction
- Quantum Rényi entropies: background and properties
- The Hadamard three-line theorem
- The proof

# Rényi entropy

Rényi entropy of order  $\alpha$ :

$$H_\alpha(X) = \frac{1}{1-\alpha} \log \left( \sum_x p_X(x)^\alpha \right)$$

Quantum version:

$$H_\alpha(A)_\rho = \frac{1}{1-\alpha} \log \text{Tr}[\rho_A^\alpha]$$

## Rényi divergence

$$D_\alpha(p\|q) = \frac{1}{\alpha - 1} \log \left( \sum_x p(x)^\alpha q(x)^{1-\alpha} \right)$$

We can use this to define a conditional Rényi entropy:

$$\begin{aligned} H_\alpha(X|Y)_p &= - \inf_{q_Y} D_\alpha(p_{XY} \| \mathbb{1}_X \otimes q_Y) \\ &= \sup_{q_Y} \frac{1}{1 - \alpha} \log \left( \sum_{xy} p_{XY}(x, y)^\alpha q_Y(y)^{1-\alpha} \right). \end{aligned}$$

How do we get a quantum version?

# Rényi divergence

If  $\rho$  and  $\sigma$  don't commute, many possible definitions:

$$D_\alpha(\rho\|\sigma) = \frac{1}{\alpha - 1} \log \text{Tr} [\rho^\alpha \sigma^{1-\alpha}]$$

$$D_\alpha(\rho\|\sigma) = \frac{1}{\alpha - 1} \log \text{Tr} \left[ \rho^{\frac{\alpha}{2}} \sigma^{\frac{1-\alpha}{2}} \rho^{\frac{\alpha}{2}} \sigma^{\frac{1-\alpha}{2}} \right]$$

$$D_\alpha(\rho\|\sigma) = \frac{1}{\alpha - 1} \log \text{Tr} \left[ \left( \sigma^{\frac{1-\alpha}{2\alpha}} \rho \sigma^{\frac{1-\alpha}{2\alpha}} \right)^\alpha \right]$$

⋮

How to pick the right one?

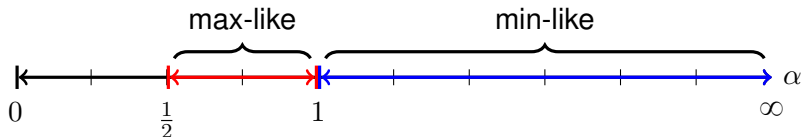
## “Sandwiched” Rényi entropy

$$H_\alpha(A|B)_\rho = -\inf_{\sigma_B} D_\alpha(\rho_{AB} \| \mathbb{1}_A \otimes \sigma_B)$$

### Properties:

- $\lim_{\alpha \rightarrow \infty} H_\alpha(A|B)_\rho = H_{\min}(A|B)_\rho$
- $\lim_{\alpha \rightarrow 1} H_\alpha(A|B)_\rho = H(A|B)_\rho$
- $H_{\frac{1}{2}}(A|B)_\rho = H_{\max}(A|B)_\rho$
- Monotone decreasing in  $\alpha$ : if  $\frac{1}{2} \leq \alpha \leq \gamma$ , then  $H_\alpha(A|B)_\rho \geq H_\gamma(A|B)_\rho$
- Data processing:  $H_\alpha(A|BC)_\rho \leq H_\alpha(A|B)_\rho$  for  $\alpha \geq \frac{1}{2}$ .
- Duality: for pure  $\psi_{ABC}$ ,  $H_\alpha(A|B)_\psi = -H_\gamma(A|C)_\psi$ , where  $\frac{1}{\alpha} + \frac{1}{\gamma} = 2$ .

## Sandwiched Rényi entropy: range of $\alpha$



- $\alpha < 1$ : “max-like” entropies: sensitive to large numbers of small eigenvalues
- $\alpha > 1$ : “min-like” entropies: sensitive to small numbers of large eigenvalues
- Duality: for pure  $\psi_{ABC}$ ,  $H_\alpha(A|B)_\psi = -H_\gamma(A|C)_\psi$ , with  $\frac{1}{\alpha} + \frac{1}{\gamma} = 2$ .

# Sandwiched Rényi entropy: applications

- Strong converses:
  - M. M. Wilde, A. Winter, and D. Yang, arXiv: 1306.1586
  - T. Cooney, M. Mosonyi, and M. M. Wilde, arXiv: 1408.3373
  - M. Tomamichel, M. M. Wilde, and A. Winter, arXiv: 1406.2946
- Hypothesis testing
  - M. Mosonyi and T. Ogawa, arXiv: 1309.3228
  - M. Mosonyi and T. Ogawa, arXiv: 1409.3562
- ...



# Chain rules

Goal of this talk: prove chain rules for sandwiched Rényi entropies.

Given a state  $\rho_{ABC}$  and  $\alpha, \beta, \gamma$  with  $\frac{\alpha}{\alpha-1} = \frac{\beta}{\beta-1} + \frac{\gamma}{\gamma-1}$ :

$$H_{\alpha}(AB|C)_{\rho} \geq H_{\beta}(A|BC)_{\rho} + H_{\gamma}(B|C)_{\rho}$$

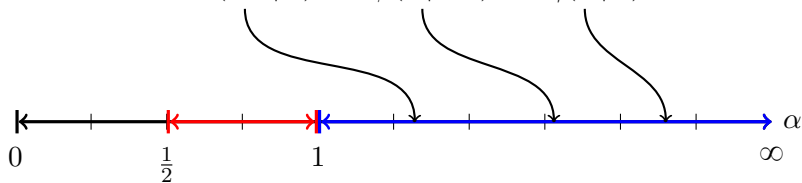
if  $\alpha, \beta, \gamma > 1$ , and

$$H_{\alpha}(AB|C)_{\rho} \leq H_{\beta}(A|BC)_{\rho} + H_{\gamma}(B|C)_{\rho}$$

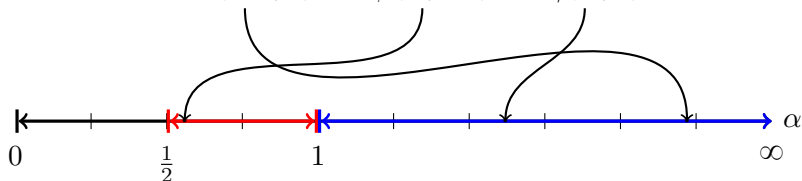
if  $\alpha > 1$ , and one of  $\beta, \gamma$  is  $< 1$ .

## Chain rules: parameters

$$H_\alpha(AB|C) \geq H_\beta(A|BC) + H_\gamma(B|C)$$



$$H_\alpha(AB|C) \leq H_\beta(A|BC) + H_\gamma(B|C)$$



# The proof

- Graphical calculus and expressions for  $H_\alpha$  (+bonus proof of duality)
- Hadamard three-line theorem (+bonus proof of monotonicity in  $\alpha$ )
- Actual proof

# Graphical calculus

$$|\psi\rangle = \text{---} \text{---} \left[ \begin{array}{c} \psi \end{array} \right]$$

$$|\psi\rangle_{AB} = \text{---} \left[ \begin{array}{c} A \\ \psi \\ B \end{array} \right]$$

$$\sigma = \text{---} \left[ \begin{array}{c} \sigma \end{array} \right] \text{---}$$

$$U = \text{---} \left[ \begin{array}{c} U \end{array} \right] \text{---}$$

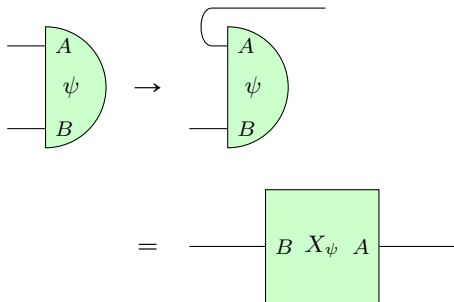
# Graphical calculus

Turning kets into bras:

$$\begin{aligned} |\psi\rangle &= \sum_i a_i |i\rangle \\ &= \text{---} \text{---} \left[ \text{---} \right] \psi \\ &\rightarrow \text{---} \left[ \text{---} \right] \psi \\ &= \sum_i a_i \langle i|. \end{aligned}$$

# Graphical calculus

$$|\psi\rangle_{AB} = \sum_{ij} \lambda_{ij} |i\rangle_A \otimes |j\rangle_B \rightarrow X_\psi = \sum_{ij} \lambda_{ij} |j\rangle_B \langle i|_A$$



# Graphical calculus: trace

$$\text{Tr}[\rho] = \text{Diagram of a box labeled } \rho \text{ with a wire looping back to its top, representing the trace operation.}$$

$$\text{Tr}_A[|\psi\rangle\langle\psi|_{AB}] = \text{Diagram of two semi-circular boxes labeled } \psi \text{ on system } B, \text{ one on the left and one on the right. The top halves are connected by a wire, and the bottom halves have wires extending outwards. The top half of the left box is labeled } A \text{ and the bottom half is labeled } B. \text{ The top half of the right box is labeled } A \text{ and the bottom half is labeled } B.$$

$$= \text{Diagram of two square boxes. The left box is labeled } B X_\psi A \text{ and the right box is labeled } A X_\psi^\dagger B. \text{ They are connected by a horizontal wire. The left box has a wire entering from the left labeled } B \text{ and a wire exiting to the right labeled } A. \text{ The right box has a wire entering from the left labeled } A \text{ and a wire exiting to the right labeled } B.$$

# Schatten $p$ -norms

Definition:

$$\begin{aligned}\|X\|_p &:= \operatorname{Tr} [|X|^p]^{1/p} \\ &= \operatorname{Tr} [(X X^\dagger)^{p/2}]^{1/p}\end{aligned}$$

Dual norm: let  $\frac{1}{p} + \frac{1}{q} = 1$ .

$$\begin{aligned}\|X_{A \rightarrow B}\|_p &= \sup_{Y_{A \rightarrow B}: \|Y\|_q \leq 1} \operatorname{Tr} [Y^\dagger X] \\ &= \sup_{\sigma \geq 0: \operatorname{Tr}[\sigma] \leq 1} \operatorname{Tr} [\sigma]\end{aligned}$$

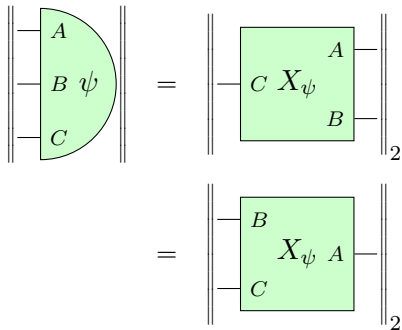


Lemma: If  $\alpha' = 1 - \frac{1}{\alpha}$ , we have that:

$$\begin{aligned}
 \left\| \left\| \begin{array}{c} B \\ X_\psi \\ A \end{array} \right\| \right\|_{2\alpha}^{\frac{1}{\alpha'}} &= \sup_{\sigma} \left\| \left\| \begin{array}{c} \sigma^{\frac{\alpha'}{2}} \\ B \\ X_\psi \\ A \end{array} \right\| \right\|_{2}^{\frac{1}{\alpha'}} \\
 &= \sup_{\sigma} \left\| \left\| \begin{array}{c} B \\ X_\psi \\ A \\ \sigma^{\frac{\alpha'}{2}} \end{array} \right\| \right\|_{2}^{\frac{1}{\alpha'}}
 \end{aligned}$$

We will be doing lots of this.

Another useful fact:



Let  $\psi_{ABC}$  be pure. Then,

$$\begin{aligned}
 H_\alpha(A|B)_\psi &= \sup_\sigma \frac{\alpha}{1-\alpha} \log \text{Tr} \left[ \left( \sigma_B^{-\frac{\alpha'}{2}} \psi_{AB} \sigma_B^{-\frac{\alpha'}{2}} \right)^\alpha \right]^{\frac{1}{\alpha}} \\
 &= \sup_\sigma \frac{-1}{\alpha'} \log \left\| \sigma_B^{-\frac{\alpha'}{2}} \psi_{AB} \sigma_B^{-\frac{\alpha'}{2}} \right\|_\alpha \\
 &= \sup_\sigma \frac{-1}{\alpha'} \log \left\| \begin{array}{c} \text{---} \\ \text{---} \sigma^{-\frac{\alpha'}{2}} \text{---} \\ \text{---} \end{array} \begin{array}{c} \text{A} \\ \text{B} \psi \\ \text{C} \end{array} \begin{array}{c} \text{A} \\ \text{C} \end{array} \begin{array}{c} \text{---} \\ \text{---} \psi \text{---} \\ \text{---} \end{array} \begin{array}{c} \text{A} \\ \text{C} \end{array} \sigma^{-\frac{\alpha'}{2}} \text{---} \right\|_\alpha \\
 &= \sup_\sigma \frac{-1}{\alpha'} \log \left\| \begin{array}{c} \text{---} \\ \text{---} \sigma^{-\frac{\alpha'}{2}} \text{---} \\ \text{---} \end{array} \begin{array}{c} \text{A} \\ \text{B} \\ \text{C} \end{array} \right\|_{2\alpha} \\
 &= \sup_\sigma \log \left\| \begin{array}{c} \text{---} \\ \text{---} \sigma^{-\frac{\alpha'}{2}} \text{---} \\ \text{---} \end{array} \begin{array}{c} \text{A} \\ \text{B} \\ \text{C} \end{array} \right\|_{\frac{2}{\alpha'}}
 \end{aligned}$$

$$\begin{aligned}
H_\alpha(A|B)_\psi &= \sup_\sigma \log \left\| \begin{array}{c} \text{---} \\ \text{---} \end{array} \begin{array}{|c|} \hline \sigma^{-\frac{\alpha'}{2}} \\ \hline \end{array} \begin{array}{|c|} \hline A \\ B \\ C \\ \hline \end{array} \right\|_{2\alpha}^{\frac{-2}{\alpha'}} \\
&= \sup_\sigma \inf_\tau \log \left\| \begin{array}{c} \text{---} \\ \text{---} \end{array} \begin{array}{|c|} \hline \sigma^{-\frac{\alpha'}{2}} \\ \hline \end{array} \begin{array}{|c|} \hline A \\ B \\ C \\ \hline \end{array} \begin{array}{|c|} \hline \tau^{\frac{\alpha'}{2}} \\ \hline \end{array} \right\|_{2}^{\frac{-2}{\alpha'}} \\
&= \inf_\tau \sup_\sigma \log \left\| \begin{array}{c} \text{---} \\ \text{---} \end{array} \begin{array}{|c|} \hline \sigma^{-\frac{\alpha'}{2}} \\ \hline \end{array} \begin{array}{|c|} \hline A \\ B \\ C \\ \hline \end{array} \begin{array}{|c|} \hline \tau^{\frac{\alpha'}{2}} \\ \hline \end{array} \right\|_{2}^{\frac{-2}{\alpha'}}
\end{aligned}$$

# First benefit: duality

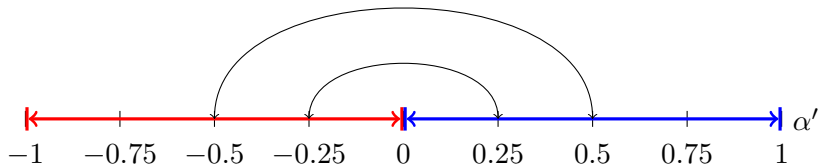
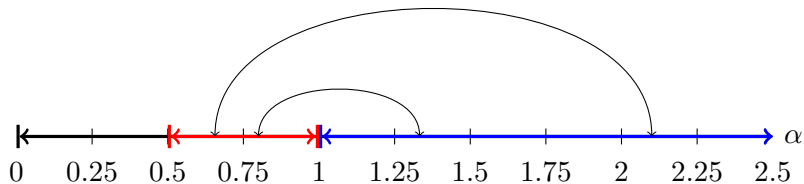
Let  $\alpha$  and  $\beta$  be such that  $\frac{1}{\alpha} + \frac{1}{\beta} = 2$ . Then,  $\beta' = -\alpha'$ . We have

$$\begin{aligned}
 H_\alpha(A|B)_\psi &= \log \sup_\sigma \inf_\tau \left\| \begin{array}{c} \text{---} \\ \text{---} \end{array} \right\|_{\frac{-2}{\alpha'}} \left\| \begin{array}{c} \text{---} \\ \text{---} \end{array} \right\|_2 \left\| \begin{array}{c} \text{---} \\ \text{---} \end{array} \right\|_{\frac{-2}{\alpha'}} \\
 &= \log \sup_\sigma \inf_\tau \left\| \begin{array}{c} \text{---} \\ \text{---} \end{array} \right\|_{\frac{2}{\beta'}} \left\| \begin{array}{c} \text{---} \\ \text{---} \end{array} \right\|_2 \left\| \begin{array}{c} \text{---} \\ \text{---} \end{array} \right\|_{\frac{2}{\beta'}} \\
 &= \log \left( \inf_\sigma \sup_\tau \left\| \begin{array}{c} \text{---} \\ \text{---} \end{array} \right\|_{\frac{2}{\beta'}} \left\| \begin{array}{c} \text{---} \\ \text{---} \end{array} \right\|_2 \left\| \begin{array}{c} \text{---} \\ \text{---} \end{array} \right\|_{\frac{-2}{\beta'}} \right)^{-1} \\
 &= -H_\beta(A|C)_\psi
 \end{aligned}$$

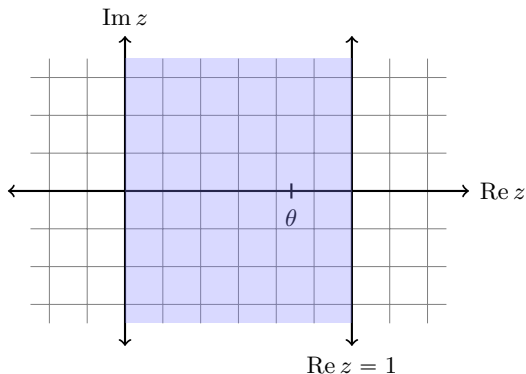
# Duality

$$H_\alpha(A|B)_\psi = -H_\beta(A|C)_\psi,$$

with  $\frac{1}{\alpha} + \frac{1}{\beta} = 2 \Rightarrow \beta' = -\alpha'$ .

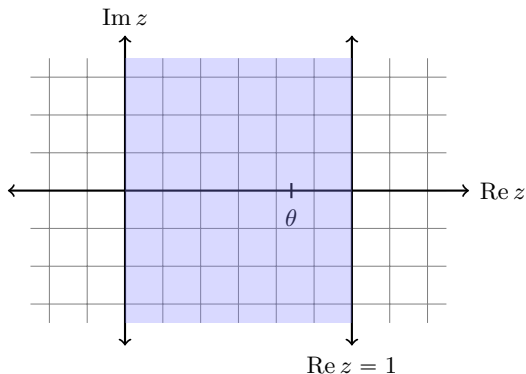


# Hadamard's three-line theorem



- $F(z)$ : holomorphic on interior of blue area, continuous at boundary
- Theorem:  $|F(\theta)| \leq (\sup_{t \in \mathbb{R}} |F(it)|)^{1-\theta} (\sup_{t \in \mathbb{R}} |F(1 + it)|)^\theta$
- Proof: maximum modulus principle

# Hadamard's three-line theorem



- $F(z) : \mathbb{C} \rightarrow \mathbb{C}^{m \times n}$ : holomorphic on interior, continuous
- Theorem:  $\|F(\theta)\|_{p_\theta} \leq (\sup_t \|F(it)\|_{p_0})^{1-\theta} (\sup_t \|F(1+it)\|_{p_1})^\theta$
- $\frac{1}{p_\theta} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$



# Monotonicity in $\alpha$

Let  $1 < \alpha < \gamma$ . We want to prove that  $H_\alpha(A|B)_\psi \geq H_\gamma(A|B)_\psi$ .

$$\begin{aligned}
 1 &= \left\| \begin{array}{c} \text{---} \\ \sigma^0 \\ \text{---} \end{array} \begin{array}{c} A \\ B \\ C \end{array} \right\|_2 \\
 2^{-\frac{\alpha'}{2}} H_\alpha(A|B)_{\rho|\sigma} &= \left\| \begin{array}{c} \text{---} \\ \sigma^{-\frac{\alpha'}{2}} \\ \text{---} \end{array} \begin{array}{c} A \\ B \\ C \end{array} \right\|_{2\alpha} \\
 2^{-\frac{\gamma'}{2}} H_\gamma(A|B)_{\rho|\sigma} &= \left\| \begin{array}{c} \text{---} \\ \sigma^{-\frac{\gamma'}{2}} \\ \text{---} \end{array} \begin{array}{c} A \\ B \\ C \end{array} \right\|_{2\gamma}
 \end{aligned}$$

Choose:  $F(z) = \sigma^{-\frac{z\gamma'}{2}} X$        $p_0 = 2$        $p_1 = 2\gamma$        $\theta = \frac{\alpha'}{\gamma'}$

$$\begin{aligned}
 F(z) &= \text{Diagram} \\
 p_0 &= 2 \\
 p_1 &= 2\gamma \\
 \theta &= \frac{\alpha'}{\gamma'}
 \end{aligned}$$

We can calculate that  $p_\theta = 2\alpha$ , and  $\|F(\theta)\|_{2\alpha} = 2^{\frac{-\alpha'}{2}} H_\alpha(A|B)_{\rho|\sigma}$ .

$$F(z) = \begin{array}{c} \text{---} \\ \text{---} \end{array} \begin{array}{c} \boxed{\sigma \frac{-z\gamma'}{2}} \\ \boxed{\begin{array}{cc} A & \\ B & C \end{array}} \end{array} \text{---}$$

From the three-line theorem:

$$\|F(\theta)\|_{2\alpha} \leq \left( \sup_{t \in \mathbb{R}} \|F(it)\|_2 \right)^{1 - \frac{\alpha'}{\gamma'}} \left( \sup_{t \in \mathbb{R}} \|F(1 + it)\|_{2\gamma} \right)^{\frac{\alpha'}{\gamma'}}$$

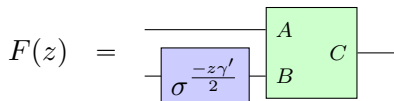
$$\|F(it)\|_2 = \left\| \begin{array}{c} \text{---} \\ \text{---} \end{array} \begin{array}{c} \boxed{\sigma \frac{-it\gamma'}{2}} \\ \boxed{\begin{array}{cc} A & \\ B & C \end{array}} \end{array} \right\|_2$$

$$F(z) = \begin{array}{c} \text{---} \\ \text{---} \end{array} \begin{array}{|c|} \hline A \\ \hline B \\ \hline \end{array} \begin{array}{c} \text{---} \\ \text{---} \end{array} \\ \begin{array}{|c|} \hline \sigma \frac{-z\gamma'}{2} \\ \hline \end{array} \begin{array}{|c|} \hline A \\ \hline B \\ \hline \end{array} \begin{array}{c} \text{---} \\ \text{---} \end{array} \quad C$$

From the three-line theorem:

$$\|F(\theta)\|_{2\alpha} \leq \left( \sup_{t \in \mathbb{R}} \|F(it)\|_2 \right)^{1 - \frac{\alpha'}{\gamma'}} \left( \sup_{t \in \mathbb{R}} \|F(1 + it)\|_{2\gamma} \right)^{\frac{\alpha'}{\gamma'}}$$

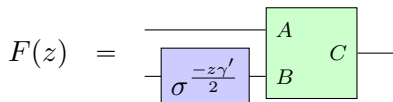
$$\begin{aligned} \|F(it)\|_2 &= \left\| \begin{array}{c} \text{---} \\ \text{---} \end{array} \begin{array}{|c|} \hline A \\ \hline B \\ \hline \end{array} \begin{array}{c} \text{---} \\ \text{---} \end{array} \right\|_2 \\ &= \left\| \begin{array}{|c|} \hline A \\ \hline B \\ \hline \end{array} \begin{array}{c} \text{---} \\ \text{---} \end{array} \right\|_2 = 1 \end{aligned}$$



From the three-line theorem:

$$\|F(\theta)\|_{2\alpha} \leq \left( \sup_{t \in \mathbb{R}} \|F(it)\|_2 \right)^{1 - \frac{\alpha'}{\gamma'}} \left( \sup_{t \in \mathbb{R}} \|F(1 + it)\|_{2\gamma} \right)^{\frac{\alpha'}{\gamma'}}$$

$$\|F(1 + it)\|_{2\gamma} = \left\| \begin{array}{c} \text{---} \\ \text{---} \end{array} \begin{array}{c} A \\ B \end{array} \begin{array}{c} C \\ \text{---} \end{array} \right\|_{2\gamma}$$



From the three-line theorem:

$$\|F(\theta)\|_{2\alpha} \leq \left( \sup_{t \in \mathbb{R}} \|F(it)\|_2 \right)^{1 - \frac{\alpha'}{\gamma'}} \left( \sup_{t \in \mathbb{R}} \|F(1 + it)\|_{2\gamma} \right)^{\frac{\alpha'}{\gamma'}}$$

$$\begin{aligned} \|F(1 + it)\|_{2\gamma} &= \left\| \begin{array}{c} \text{---} \\ \text{---} \end{array} \begin{array}{|c|} \hline \sigma \frac{-it\gamma'}{2} \\ \hline \sigma \frac{-\gamma'}{2} \\ \hline \end{array} \begin{array}{|c|} \hline A \\ \hline B \\ \hline \end{array} \begin{array}{c} \text{---} \\ \text{---} \end{array} \right\|_{2\gamma} \\ &= \left\| \begin{array}{c} \text{---} \\ \text{---} \end{array} \begin{array}{|c|} \hline \sigma \frac{-\gamma'}{2} \\ \hline \sigma \frac{-\gamma'}{2} \\ \hline \end{array} \begin{array}{|c|} \hline A \\ \hline B \\ \hline \end{array} \begin{array}{c} \text{---} \\ \text{---} \end{array} \right\|_{2\gamma} = 2^{\frac{-\gamma'}{2}} H_\gamma(A|B)_{\rho|\sigma} \end{aligned}$$

Putting it all together:

$$\begin{aligned} 2^{-\frac{\alpha'}{2}} H_{\alpha}(A|B)_{\rho|\sigma} &= \|F(\theta)\|_{2\alpha} \\ &\leq \left( 2^{-\frac{\gamma'}{2}} H_{\gamma}(A|B)_{\rho|\sigma} \right)^{\frac{\alpha'}{\gamma}} \\ &= 2^{-\frac{\alpha'}{2}} H_{\gamma}(A|B)_{\rho|\sigma} \end{aligned}$$

and therefore

$$\begin{aligned} H_{\alpha}(A|B)_{\rho|\sigma} &\geq H_{\gamma}(A|B)_{\rho|\sigma} \\ H_{\alpha}(A|B)_{\rho} &\geq H_{\gamma}(A|B)_{\rho}. \end{aligned}$$

We can use slightly different parameter choices to show this for  $\frac{1}{2} \leq \alpha \leq \gamma$ .

# Chain rules

And now for the main event: we will prove that

$$H_{\alpha}(AB|C) \geq H_{\beta}(A|BC) + H_{\gamma}(B|C)$$

with  $\alpha, \beta, \gamma > 1$  and

$$\frac{1}{\alpha'} = \frac{1}{\beta'} + \frac{1}{\gamma'}$$



$$2^{\frac{-\alpha'}{2}} H_{\alpha}(AB|C)_{\rho|\sigma} = \sup_{\tau} \left\| \begin{array}{c} \text{---} \\ \text{---} \end{array} \right\|_2$$

The diagram shows a quantum circuit with two horizontal lines. A central green box labeled  $X_{\psi}$  has inputs  $B$  and  $C$  on the top line and  $A$  and  $D$  on the bottom line. To the left of the green box is a blue box containing  $\sigma^{\frac{-\alpha'}{2}}$  on the bottom line. To the right of the green box is a blue box containing  $\tau^{\frac{\alpha'}{2}}$  on the bottom line. The entire circuit is enclosed in double vertical bars, with a subscript  $2$  at the bottom right.

$$2^{\frac{-\beta'}{2}} H_{\beta}(A|BC)_{\rho} = \sup_{\tau} \left\| \begin{array}{c} \text{---} \\ \text{---} \end{array} \right\|_{2^{\hat{\beta}}}$$

The diagram shows a quantum circuit with two horizontal lines. A central green box labeled  $X_{\psi}$  has inputs  $B$  and  $C$  on the top line and  $A$  and  $D$  on the bottom line. To the right of the green box is a blue box containing  $\tau^{\frac{\beta'}{2}}$  on the bottom line. The entire circuit is enclosed in double vertical bars, with a subscript  $2^{\hat{\beta}}$  at the bottom right.

$$2^{\frac{-\gamma'}{2}} H_{\gamma}(B|C)_{\rho|\sigma} = \left\| \begin{array}{c} \text{---} \\ \text{---} \end{array} \right\|_{2^{\gamma}}$$

The diagram shows a quantum circuit with two horizontal lines. A central green box labeled  $X_{\psi}$  has inputs  $B$  and  $C$  on the top line and  $A$  and  $D$  on the bottom line. To the left of the green box is a blue box containing  $\sigma^{\frac{-\gamma'}{2}}$  on the bottom line. The entire circuit is enclosed in double vertical bars, with a subscript  $2^{\gamma}$  at the bottom right.

$$2^{\frac{-\alpha'}{2}} H_{\alpha}(AB|C)_{\rho|\sigma} = \sup_{\tau} \left\| \begin{array}{c} \text{---} \\ \text{---} \end{array} \right\|_2$$

$$2^{\frac{-\alpha'}{2}} H_{\beta}(A|BC)_{\rho} = \sup_{\tau} \left\| \begin{array}{c} \text{---} \\ \text{---} \end{array} \right\|_{2\hat{\beta}}$$

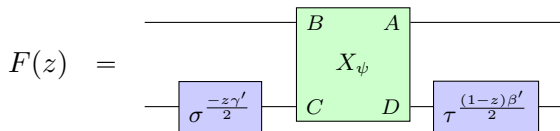
$$2^{\frac{-\alpha'}{2}} H_{\gamma}(B|C)_{\rho|\sigma} = \left\| \begin{array}{c} \text{---} \\ \text{---} \end{array} \right\|_{2\gamma}$$

Choose

$$F(z) = \sigma^{\frac{-z\gamma'}{2}} X_{\psi} \tau^{\frac{(1-z)\beta'}{2}}$$

$$\theta = \frac{\alpha'}{\gamma'}$$

$$1 - \theta = \frac{\alpha'}{\beta'}$$



$$\theta = \frac{\alpha'}{\gamma'}$$

$$1 - \theta = \frac{\alpha'}{\beta'}$$

$$p_0 = 2\hat{\beta}$$

$$p_1 = 2\gamma$$

This is true if  $\frac{\alpha'}{\beta'} + \frac{\alpha'}{\gamma'} = 1$ , so  $\frac{1}{\alpha'} = \frac{1}{\beta'} + \frac{1}{\gamma'}$ .

Recall:

$$F(z) = \begin{array}{c} \text{---} \\ \text{---} \end{array} \begin{array}{|c|c|} \hline B & A \\ \hline X_\psi \\ \hline C & D \\ \hline \end{array} \begin{array}{c} \text{---} \\ \text{---} \end{array}$$

$\left[ \sigma \frac{-z\gamma'}{2} \right] \quad \left[ \tau \frac{(1-z)\beta'}{2} \right]$

From the 3-line theorem:

$$\|F(\theta)\|_{p_\theta} \leq \left( \sup_t \|F(it)\|_{p_0} \right)^{1-\theta} \left( \sup_t \|F(1+it)\|_{p_1} \right)^\theta$$

$$\|F(it)\|_{p_0}^{1-\theta} = \left\| \begin{array}{c} \text{---} \\ \text{---} \end{array} \begin{array}{|c|c|} \hline B & A \\ \hline X_\psi \\ \hline C & D \\ \hline \end{array} \begin{array}{c} \text{---} \\ \text{---} \end{array} \right\|_{2\hat{\beta}}^{\frac{\alpha'}{\beta'}}$$

$\left[ \sigma \frac{-it\gamma'}{2} \right] \quad \left[ \tau \frac{(1-it)\beta'}{2} \right]$

Recall:

$$F(z) = \begin{array}{c} \text{---} \\ \text{---} \end{array} \begin{array}{c} B \quad A \\ X_\psi \\ C \quad D \end{array} \begin{array}{c} \text{---} \\ \text{---} \end{array} \\ \left[ \sigma \frac{-z\gamma'}{2} \right] \quad \left[ \tau \frac{(1-z)\beta'}{2} \right]$$

From the 3-line theorem:

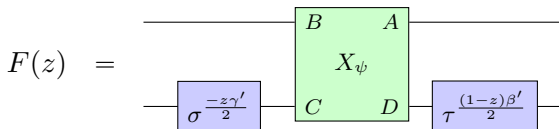
$$\|F(\theta)\|_{p_\theta} \leq \left( \sup_t \|F(it)\|_{p_0} \right)^{1-\theta} \left( \sup_t \|F(1+it)\|_{p_1} \right)^\theta$$

$$\|F(it)\|_{p_0}^{1-\theta} = \left\| \begin{array}{c} \text{---} \\ \text{---} \end{array} \begin{array}{c} B \quad A \\ X_\psi \\ C \quad D \end{array} \begin{array}{c} \text{---} \\ \text{---} \end{array} \right\|_{2\hat{\beta}}^{\frac{\alpha'}{\beta'}}$$

$\frac{\alpha'}{\beta'}$

$2\hat{\beta}$

Recall:



From the 3-line theorem:

$$\|F(\theta)\|_{p_\theta} \leq \left( \sup_t \|F(it)\|_{p_0} \right)^{1-\theta} \left( \sup_t \|F(1+it)\|_{p_1} \right)^\theta$$

$$\|F(it)\|_{p_0}^{1-\theta} = \left\| \left\| \begin{array}{c} \text{---} \\ \text{---} \end{array} \right\|_{\frac{\alpha'}{\beta'}} \begin{array}{c} B \quad A \\ X_\psi \\ C \quad D \end{array} \begin{array}{c} \text{---} \\ \text{---} \end{array} \left\|_{2\hat{\beta}} \right\|_{\tau \frac{\beta'}{2}}$$

$$= \|F(0)\|_{p_0}^{1-\theta} = 2^{\frac{-\alpha'}{2}} H_\beta(A|BC)_\rho$$

Recall:

$$F(z) = \begin{array}{c} \text{---} \\ \text{---} \end{array} \begin{array}{|c|c|} \hline B & A \\ \hline X_\psi & \\ \hline C & D \\ \hline \end{array} \begin{array}{c} \text{---} \\ \text{---} \end{array} \\ \begin{array}{|c|} \hline \sigma \frac{-z\gamma'}{2} \\ \hline \end{array} \begin{array}{|c|c|} \hline C & D \\ \hline \end{array} \begin{array}{|c|} \hline \tau \frac{(1-z)\beta'}{2} \\ \hline \end{array}$$

From the 3-line theorem:

$$\|F(\theta)\|_{p_\theta} \leq \left( \sup_t \|F(it)\|_{p_0} \right)^{1-\theta} \left( \sup_t \|F(1+it)\|_{p_1} \right)^\theta$$

$$\|F(1+it)\|_{p_1}^\theta = \left\| \begin{array}{c} \text{---} \\ \text{---} \end{array} \begin{array}{|c|c|} \hline B & A \\ \hline X_\psi & \\ \hline C & D \\ \hline \end{array} \begin{array}{c} \text{---} \\ \text{---} \end{array} \right\|_{2\gamma}^{\frac{\alpha'}{\gamma}}$$

$\begin{array}{|c|} \hline \sigma \frac{-(1+it)\gamma'}{2} \\ \hline \end{array} \begin{array}{|c|c|} \hline C & D \\ \hline \end{array} \begin{array}{|c|} \hline \tau \frac{-it\beta'}{2} \\ \hline \end{array}$

Recall:

$$F(z) = \begin{array}{c} \text{---} \\ \text{---} \end{array} \begin{array}{c} B \quad A \\ X_\psi \\ C \quad D \end{array} \begin{array}{c} \text{---} \\ \text{---} \end{array} \\ \left[ \sigma \frac{-z\gamma'}{2} \right] \quad \left[ \tau \frac{(1-z)\beta'}{2} \right]$$

From the 3-line theorem:

$$\|F(\theta)\|_{p_\theta} \leq \left( \sup_t \|F(it)\|_{p_0} \right)^{1-\theta} \left( \sup_t \|F(1+it)\|_{p_1} \right)^\theta$$

$$\|F(1+it)\|_{p_1}^\theta = \left\| \begin{array}{c} \text{---} \\ \text{---} \end{array} \begin{array}{c} B \quad A \\ X_\psi \\ C \quad D \end{array} \begin{array}{c} \text{---} \\ \text{---} \end{array} \right\|_{\frac{p_1}{\theta}} \\ \left[ \sigma \frac{-it\gamma'}{2} \right] \left[ \sigma \frac{-\gamma'}{2} \right] \quad \left[ \tau \frac{-it\beta'}{2} \right]$$



Recall:

$$F(z) = \begin{array}{c} \text{---} \\ \text{---} \end{array} \begin{array}{c} B \quad A \\ X_\psi \\ C \quad D \end{array} \begin{array}{c} \text{---} \\ \text{---} \end{array} \\ \begin{array}{c} \sigma \frac{-z\gamma'}{2} \\ \tau \frac{(1-z)\beta'}{2} \end{array}$$

From the 3-line theorem:

$$\|F(\theta)\|_{p_\theta} \leq \left( \sup_t \|F(it)\|_{p_0} \right)^{1-\theta} \left( \sup_t \|F(1+it)\|_{p_1} \right)^\theta$$

$$\|F(1+it)\|_{p_1}^\theta = \left\| \begin{array}{c} \text{---} \\ \text{---} \end{array} \begin{array}{c} B \quad A \\ X_\psi \\ C \quad D \end{array} \begin{array}{c} \text{---} \\ \text{---} \end{array} \right\|_{2\gamma}^{\frac{\alpha'}{\gamma}}$$

$$= \|F(1)\|_{p_1}^\theta = 2^{\frac{-\alpha'}{2}} H_\gamma(B|C)_{\rho|\sigma}$$

So we've proven that

$$H_\alpha(AB|C) \geq H_\beta(A|BC) + H_\gamma(B|C)$$

for  $\alpha, \beta, \gamma > 1$ , where

$$\frac{1}{\alpha'} = \frac{1}{\beta'} + \frac{1}{\gamma'}.$$

By duality, if  $\alpha, \beta, \gamma < 1$ , we have that

$$H_\alpha(AB|C) \leq H_\beta(A|BC) + H_\gamma(B|C).$$

How about the other cases?

## Chain rules: the second case

Now, suppose that  $\alpha > 1$ , and one of  $\beta, \gamma$  is below 1. We will show that

$$H_\alpha(AB|C) \leq H_\beta(A|BC) + H_\gamma(B|C),$$

with

$$\frac{1}{\alpha'} = \frac{1}{\beta'} + \frac{1}{\gamma'}$$

as before.

Goal:  $H_\beta(A|BC) \geq H_\alpha(AB|C) - H_\gamma(B|C)$

$$2^{\frac{-\alpha'}{2}} H_\alpha(AB|C)_{\rho|\sigma} = \sup_{\tau} \left\| \begin{array}{c} \text{---} \\ \text{---} \\ \text{---} \\ \text{---} \end{array} \begin{array}{c} B \quad A \\ X_\psi \\ C \quad D \end{array} \begin{array}{c} \text{---} \\ \text{---} \\ \text{---} \\ \text{---} \end{array} \right\|_2$$

$\sigma^{\frac{-\alpha'}{2}}$ 
 $\tau^{\frac{\alpha'}{2}}$

$$2^{\frac{-\beta'}{2}} H_\beta(A|BC)_\rho = \sup_{\tau} \left\| \begin{array}{c} \text{---} \\ \text{---} \\ \text{---} \\ \text{---} \end{array} \begin{array}{c} B \quad A \\ X_\psi \\ C \quad D \end{array} \begin{array}{c} \text{---} \\ \text{---} \\ \text{---} \\ \text{---} \end{array} \right\|_{2^{\hat{\beta}}}$$

$\tau^{\frac{\beta'}{2}}$

$$2^{\frac{-\gamma'}{2}} H_\gamma(B|C)_{\rho|\sigma} = \left\| \begin{array}{c} \text{---} \\ \text{---} \\ \text{---} \\ \text{---} \end{array} \begin{array}{c} B \quad A \\ X_\psi \\ C \quad D \end{array} \begin{array}{c} \text{---} \\ \text{---} \\ \text{---} \\ \text{---} \end{array} \right\|_{2^\gamma}$$

$\sigma^{\frac{-\gamma'}{2}}$

Goal:  $H_\beta(A|BC) \geq H_\alpha(AB|C) - H_\gamma(B|C)$

$$2^{-\frac{\beta'}{2}} H_\alpha(AB|C)_{\rho|\sigma} = \sup_{\tau} \left\| \begin{array}{c} \text{---} \\ \text{---} \\ \sigma^{-\frac{\alpha'}{2}} \quad \begin{array}{|c|c|} \hline B & A \\ \hline X_\psi \\ \hline C & D \\ \hline \end{array} \quad \tau^{\frac{\alpha'}{2}} \\ \text{---} \\ \text{---} \end{array} \right\|_2^{2^{\frac{\beta'}{\alpha'}}$$

$$2^{-\frac{\beta'}{2}} H_\beta(A|BC)_\rho = \sup_{\tau} \left\| \begin{array}{c} \text{---} \\ \text{---} \\ \text{---} \\ \text{---} \end{array} \begin{array}{|c|c|} \hline B & A \\ \hline X_\psi \\ \hline C & D \\ \hline \end{array} \begin{array}{c} \text{---} \\ \text{---} \\ \tau^{\frac{\beta'}{2}} \\ \text{---} \end{array} \right\|_{2^{\hat{\beta}}}$$

$$2^{\frac{\beta'}{2}} H_\gamma(B|C)_{\rho|\sigma} = \left\| \begin{array}{c} \text{---} \\ \text{---} \\ \sigma^{-\frac{\gamma'}{2}} \quad \begin{array}{|c|c|} \hline B & A \\ \hline X_\psi \\ \hline C & D \\ \hline \end{array} \\ \text{---} \\ \text{---} \end{array} \right\|_{2^\gamma}^{2^{\frac{-\beta'}{\gamma'}}$$

Choose

$$F(z) = \sigma^{-\frac{\alpha'}{2}} \left(1 + \frac{z\gamma'}{\beta'}\right) X_\psi \tau^{\frac{\alpha'}{2}} (1-z)$$

$$\theta = \frac{-\beta'}{\gamma'}$$

$$1-\theta = \frac{\beta'}{\alpha'}$$

$$\begin{aligned}
 F(z) &= \left[ \sigma \frac{-\alpha'}{2} \left( 1 + z \frac{\gamma'}{\beta'} \right) \right] \begin{array}{|c|c|} \hline B & A \\ \hline X_\psi & \\ \hline C & D \\ \hline \end{array} \left[ \tau \frac{\alpha'}{2} (1-z) \right] \\
 \theta &= \frac{-\beta'}{\gamma'} \\
 1 - \theta &= \frac{\beta'}{\alpha'} \\
 p_0 &= 2 \\
 p_\theta &= 2\hat{\beta} \\
 p_1 &= 2\gamma
 \end{aligned}$$

This is true if  $\frac{\alpha'}{\beta'} + \frac{\alpha'}{\gamma'} = 1$ , so  $\frac{1}{\alpha'} = \frac{1}{\beta'} + \frac{1}{\gamma'}$ .

Recall:

$$F(z) = \begin{array}{c} \text{---} \\ \text{---} \end{array} \begin{array}{c} B \quad A \\ X_\psi \\ C \quad D \end{array} \begin{array}{c} \text{---} \\ \text{---} \end{array} \\ \begin{array}{c} \sigma^{-\frac{\alpha'}{2}} \left(1 + z \frac{\gamma'}{\beta'}\right) \\ \tau^{\frac{\alpha'}{2}} (1-z) \end{array}$$

From the 3-line theorem:

$$\|F(\theta)\|_{p_\theta} \leq \left( \sup_t \|F(it)\|_{p_0} \right)^{1-\theta} \left( \sup_t \|F(1+it)\|_{p_1} \right)^\theta$$

$$\|F(\theta)\|_{p_\theta} = \left\| \begin{array}{c} \text{---} \\ \text{---} \end{array} \begin{array}{c} B \quad A \\ X_\psi \\ C \quad D \end{array} \begin{array}{c} \text{---} \\ \text{---} \end{array} \right\|_2 \\ \begin{array}{c} \sigma^0 \\ \tau^{\frac{\alpha'}{2}} \left(1 + \frac{\beta'}{\gamma'}\right) \end{array}$$

Recall:

$$F(z) = \begin{array}{c} \text{---} \\ \text{---} \\ \text{---} \end{array} \begin{array}{c} B \quad A \\ X_\psi \\ C \quad D \end{array} \begin{array}{c} \text{---} \\ \text{---} \\ \text{---} \end{array}$$

$$\begin{array}{c} \text{---} \\ \text{---} \\ \text{---} \end{array} \begin{array}{c} \sigma^{-\frac{\alpha'}{2}} \left(1 + z \frac{\gamma'}{\beta'}\right) \\ \text{---} \\ \text{---} \end{array} \begin{array}{c} B \quad A \\ X_\psi \\ C \quad D \end{array} \begin{array}{c} \text{---} \\ \text{---} \\ \text{---} \end{array} \begin{array}{c} \tau^{\frac{\alpha'}{2}} (1-z) \\ \text{---} \\ \text{---} \end{array}$$

From the 3-line theorem:

$$\|F(\theta)\|_{p_\theta} \leq \left( \sup_t \|F(it)\|_{p_0} \right)^{1-\theta} \left( \sup_t \|F(1+it)\|_{p_1} \right)^\theta$$

$$\|F(\theta)\|_{p_\theta} = \left\| \begin{array}{c} \text{---} \\ \text{---} \\ \text{---} \end{array} \begin{array}{c} B \quad A \\ X_\psi \\ C \quad D \end{array} \begin{array}{c} \text{---} \\ \text{---} \\ \text{---} \end{array} \begin{array}{c} \tau^{\frac{\beta'}{2}} \\ \text{---} \\ \text{---} \end{array} \right\|_2$$

$$= 2^{-\frac{\beta'}{2}} H_\beta(A|BC)_\rho$$



Recall:

$$F(z) = \begin{array}{c} \text{---} \\ \left[ \sigma^{-\frac{\alpha'}{2}} \left( 1 + z \frac{\gamma'}{\beta'} \right) \right] \begin{array}{c} B \quad A \\ X_\psi \\ C \quad D \end{array} \left[ \tau \frac{\alpha'}{2} (1 - z) \right] \\ \text{---} \end{array}$$

From the 3-line theorem:

$$\|F(\theta)\|_{p_\theta} \leq \left( \sup_t \|F(it)\|_{p_0} \right)^{1-\theta} \left( \sup_t \|F(1+it)\|_{p_1} \right)^\theta$$

$$\|F(it)\|_{p_0}^{1-\theta} = \left\| \begin{array}{c} \text{---} \\ \left[ \sigma^{-\frac{\alpha'}{2}} \left( 1 + it \frac{\gamma'}{\beta'} \right) \right] \begin{array}{c} B \quad A \\ X_\psi \\ C \quad D \end{array} \left[ \tau \frac{\alpha'}{2} (1 - it) \right] \\ \text{---} \end{array} \right\|_{\frac{\beta'}{\alpha'}}^{1-\theta}$$

Recall:

$$F(z) = \begin{array}{c} \text{---} \\ \left[ \sigma \frac{-\alpha'}{2} \left( 1 + z \frac{\gamma'}{\beta'} \right) \right] \begin{array}{|c|c|} \hline B & A \\ \hline X_\psi & \\ \hline C & D \\ \hline \end{array} \left[ \tau \frac{\alpha'}{2} (1-z) \right] \\ \text{---} \end{array}$$

From the 3-line theorem:

$$\|F(\theta)\|_{p_\theta} \leq \left( \sup_t \|F(it)\|_{p_0} \right)^{1-\theta} \left( \sup_t \|F(1+it)\|_{p_1} \right)^\theta$$

$$\|F(it)\|_{p_0}^{1-\theta} = \left\| \begin{array}{c} \text{---} \\ \left[ \sigma \frac{-it\alpha'\gamma'}{2\beta'} \right] \left[ \sigma \frac{-\alpha'}{2} \right] \begin{array}{|c|c|} \hline B & A \\ \hline X_\psi & \\ \hline C & D \\ \hline \end{array} \left[ \tau \frac{\alpha'}{2} \right] \left[ \tau \frac{-it\alpha'}{2} \right] \\ \text{---} \end{array} \right\|_{2}^{\frac{\beta'}{\alpha'}}$$

Recall:

$$F(z) = \begin{array}{c} \text{---} \\ \text{---} \end{array} \begin{array}{c} B \quad A \\ X_\psi \\ C \quad D \end{array} \begin{array}{c} \text{---} \\ \text{---} \end{array} \\ \begin{array}{c} \sigma^{-\frac{\alpha'}{2}} (1+z\frac{\gamma'}{\beta'}) \\ \tau^{\frac{\alpha'}{2}} (1-z) \end{array} \end{array}$$

From the 3-line theorem:

$$\|F(\theta)\|_{p_\theta} \leq \left( \sup_t \|F(it)\|_{p_0} \right)^{1-\theta} \left( \sup_t \|F(1+it)\|_{p_1} \right)^\theta$$

$$\|F(it)\|_{p_0}^{1-\theta} = \left\| \begin{array}{c} \text{---} \\ \text{---} \end{array} \begin{array}{c} B \quad A \\ X_\psi \\ C \quad D \end{array} \begin{array}{c} \text{---} \\ \text{---} \end{array} \right\|_{\frac{\beta'}{\alpha'}}^{1-\theta} \\ \left\| \begin{array}{c} \sigma^{-\frac{\alpha'}{2}} \\ \tau^{\frac{\alpha'}{2}} \end{array} \right\|_2$$

$$= \|F(0)\|_{p_0}^{1-\theta} = 2^{-\frac{\beta'}{2}} H_\alpha(AB|C)_{\rho|\sigma}$$

Recall:

$$F(z) = \begin{array}{c} \text{---} \\ \left[ \sigma \frac{-\alpha'}{2} \left( 1 + z \frac{\gamma'}{\beta'} \right) \right] \begin{array}{|c|c|} \hline B & A \\ \hline X_\psi \\ \hline C & D \\ \hline \end{array} \left[ \tau \frac{\alpha'}{2} (1-z) \right] \\ \text{---} \end{array}$$

From the 3-line theorem:

$$\|F(\theta)\|_{p_\theta} \leq \left( \sup_t \|F(it)\|_{p_0} \right)^{1-\theta} \left( \sup_t \|F(1+it)\|_{p_1} \right)^\theta$$

$$\|F(1+it)\|_{p_1}^\theta = \left\| \begin{array}{c} \text{---} \\ \left[ \sigma \frac{-\alpha'}{2} \left( 1 + (1+it) \frac{\gamma'}{\beta'} \right) \right] \begin{array}{|c|c|} \hline B & A \\ \hline X_\psi \\ \hline C & D \\ \hline \end{array} \left[ \tau \frac{-it\alpha'}{2} \right] \\ \text{---} \end{array} \right\|_{2\gamma}^{\frac{-\beta'}{\gamma'}}$$

Recall:

$$F(z) = \begin{array}{c} \text{---} \\ \text{---} \\ \text{---} \end{array} \begin{array}{c} B \quad A \\ X_\psi \\ C \quad D \end{array} \begin{array}{c} \text{---} \\ \text{---} \\ \text{---} \end{array}$$

$$\begin{array}{c} \text{---} \\ \text{---} \\ \text{---} \end{array} \begin{array}{c} \sigma \frac{-\alpha'}{2} (1+z \frac{\gamma'}{\beta'}) \\ \tau \frac{\alpha'}{2} (1-z) \end{array}$$

From the 3-line theorem:

$$\|F(\theta)\|_{p_\theta} \leq \left( \sup_t \|F(it)\|_{p_0} \right)^{1-\theta} \left( \sup_t \|F(1+it)\|_{p_1} \right)^\theta$$

$$\|F(1+it)\|_{p_1}^\theta = \left\| \begin{array}{c} \text{---} \\ \text{---} \\ \text{---} \end{array} \begin{array}{c} B \quad A \\ X_\psi \\ C \quad D \end{array} \begin{array}{c} \text{---} \\ \text{---} \\ \text{---} \end{array} \right\|_{2\gamma}^{\frac{-\beta'}{\gamma'}}$$

$$\begin{array}{c} \text{---} \\ \text{---} \\ \text{---} \end{array} \begin{array}{c} \sigma \frac{-it\alpha'}{2\beta'} \\ \sigma \frac{-\gamma'}{2} \\ \tau \frac{-it\alpha'}{2} \end{array}$$

Recall:

$$F(z) = \begin{array}{c} \text{---} \\ \text{---} \end{array} \begin{array}{c} B \quad A \\ X_\psi \\ C \quad D \end{array} \begin{array}{c} \text{---} \\ \text{---} \end{array} \\ \begin{array}{c} \sigma \frac{-\alpha'}{2} (1+z \frac{\gamma'}{\beta'}) \\ \tau \frac{\alpha'}{2} (1-z) \end{array} \end{array}$$

From the 3-line theorem:

$$\|F(\theta)\|_{p_\theta} \leq \left( \sup_t \|F(it)\|_{p_0} \right)^{1-\theta} \left( \sup_t \|F(1+it)\|_{p_1} \right)^\theta$$

$$\|F(1+it)\|_{p_1}^\theta = \left\| \begin{array}{c} \text{---} \\ \text{---} \end{array} \begin{array}{c} B \quad A \\ X_\psi \\ C \quad D \end{array} \begin{array}{c} \text{---} \\ \text{---} \end{array} \right\|_{\frac{-\beta'}{\gamma'}}^{\frac{\theta}{2\gamma}}$$

$$= \|F(1)\|_{p_1}^\theta = 2^{\frac{\beta'}{2}} H_\gamma(B|C)_{\rho|\sigma}$$

So we've proven that

$$H_{\beta}(A|BC) \geq H_{\alpha}(AB|C) - H_{\gamma}(B|C),$$

or, to reformulate,

$$H_{\alpha}(AB|C) \leq H_{\beta}(A|BC) + H_{\gamma}(B|C)$$

for  $\alpha > 1$ ,  $\beta'$  and  $\gamma'$  having opposite signs, and where

$$\frac{1}{\alpha'} = \frac{1}{\beta'} + \frac{1}{\gamma'}.$$

By duality, if  $\alpha < 1$ :

$$H_{\alpha}(AB|C) \geq H_{\beta}(A|BC) + H_{\gamma}(B|C).$$

## Further work and open questions

In the same vein:

- Bounds on recoverability using the three-line theorem (Mark's talk)
- “Swiveled” Rényi entropies (arXiv: 1506.00981)

Open questions:

- Chain rules for (conditional) Rényi mutual information?



# The papers

Main result from

- F. Dupuis, “Chain rules for quantum Rényi entropies”, arXiv: 1410.5455

with material from

- S. Beigi, “Sandwiched Rényi divergence satisfies data processing inequality”, arXiv: 1306.5920
- M. Müller-Lennert, F. Dupuis, O. Szehr, S. Fehr, and M. Tomamichel, “On quantum Rényi entropies: a new generalization and some properties”, arXiv: 1306.3142
- M. M. Wilde, A. Winter, and D. Yang, “Strong converse for the classical capacity of entanglement-breaking and Hadamard channels via a sandwiched Rényi relative entropy”, arXiv: 1306.1586

The end