Trace inequalities related to quantum entropies

Rupert L. Frank
Department of Mathematics
Caltech

and

Elliott H. Lieb
Departments of Mathematics and Physics
Princeton University

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The von Neumann Entropy

For two density matrices $\rho$ and $\sigma$ (i.e., non-negative operators with trace one) their von Neumann entropy and relative entropy are defined as

$$S(\rho) = -\text{Tr} \rho \ln \rho \quad \text{and} \quad D(\rho || \sigma) = \text{Tr} \rho \ln \rho - \text{Tr} \rho \ln \sigma .$$

**Question:** How do these quantities behave under the application of quantum channels (i.e., completely positive, trace preserving linear maps)?

**A fundamental property:** SSA (Lieb, Ruskai 1973). For any density matrix $\rho_{123}$ on a tri-partite system $\mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \mathcal{H}_3$, (with notation $\rho_{12} = \text{Tr}_3 \rho_{123}$, etc.)

$$S(\rho_{12}) + S(\rho_{23}) \geq S(\rho_{123}) + S(\rho_2) \quad \equiv \quad S(\rho_{12}) + S(\rho_{23}) \geq S(\rho_1) + S(\rho_3) .$$

**Further properties:** (Lindblad 1974) Monotonicity under CPTP maps

$$D(\mathcal{E}(\rho)||\mathcal{E}(\sigma)) \leq D(\rho || \sigma) \quad \text{(MONO)}$$

Convexity of the relative entropy

$$D((1-\theta)\rho_0 + \theta \rho_1 || (1-\theta)\sigma_0 + \theta \sigma_1) \leq (1-\theta)D(\rho_0 || \sigma_0) + \theta D(\rho_1 || \sigma_1) \quad \text{(CONV)}$$
Remarkably, (SSA), (MONO) and (CONV) are equivalent (in the sense that one of these implies other ones by simple and rather abstract arguments).

**CONV** → **MONO**: (Due to Lindblad; we will use this argument later!) According to Stinespring, if \( E \) is CPTP,
\[
E(\tau) = \operatorname{Tr}_K U(\tau \otimes |\psi\rangle\langle\psi|)U^* , \quad U \text{ unitary} , \quad \psi \in K \text{ normalized} ,
\]
so with \( du = \text{Haar measure on unitaries on } K \),
\[
(d \dim K)^{-1} E(\tau) \otimes 1 = \int du (1 \otimes u)U(\tau \otimes |\psi\rangle\langle\psi|)U^*(1 \otimes u^*) .
\]

Now (CONV) implies
\[
D(E(\rho)||E(\sigma)) = D(\int du uU(\rho \otimes |\psi\rangle\langle\psi|)U^* u^* || \int du uU(\sigma \otimes |\psi\rangle\langle\psi|)U^* u^*) \\
\leq \int du D(uU(\rho \otimes |\psi\rangle\langle\psi|)U^* u^* || uU(\sigma \otimes |\psi\rangle\langle\psi|)U^* u^*) = D(\rho||\sigma)
\]

Clearly, (MONO) → (MONO'), where (MONO') is monotonicity under partial traces

**MONO’** → **CONV**: Consider block matrices \( \theta \rho^{(1)} \otimes |\uparrow\rangle\langle\uparrow| + (1-\theta) \rho^{(2)} \otimes |\downarrow\rangle\langle\downarrow| \)
Properties of the von Neumann Entropy, cont’d

Now we want to prove \((\text{MONO’}) \iff (\text{SSA})\).

Proof proceeds via another property, originally proved by Lieb, Ruskai 1973: concavity of the conditional entropy
\[ S_{1|2}(\rho_{12}) = S(\rho_{12}) - S(\rho_1) \]

\[ S_{1|2}((1 - \theta)\rho_{12} + \theta\sigma_{12}) \geq (1 - \theta)S_{1|2}(\rho_{12}) + \theta S_{1|2}(\sigma_{12}) \quad \text{(CONC)} \]

\((\text{MONO’}) \implies (\text{SSA})\) is immediate (in the right form of (SSA))

\((\text{SSA}) \implies (\text{CONC})\): Consider block matrices, as before

\((\text{CONC}) \implies (\text{MONO’})\): (due to Lieb, Ruskai) (CONC) is an equality at \(\theta = 1\), so we can differentiate there. This gives (MONO’).

This completes the circle of equivalences!

But how do we enter the circle?
How to prove SSA?

One way to prove SSA is to show that \((A, B) \mapsto \text{Tr} A \ln A - \text{Tr} A \ln B\) is convex and to use the above equivalence. (We write \((A, B)\) instead of \((\rho, \sigma)\), since no normalization on the trace is needed, only \(A > 0\) and \(B > 0\).

In fact, one has

- **Lieb concavity (1973)**
  \[(A, B) \mapsto \text{Tr} A^\alpha B^{1-\alpha} \text{ is concave for } 0 \leq \alpha \leq 1\]

- **Ando convexity (1979)**
  \[(A, B) \mapsto \text{Tr} A^\alpha B^{1-\alpha} \text{ is convex for } 1 \leq \alpha \leq 2\]

Each of these theorems implies, as \(\alpha \to 1\), convexity of \((A, B) \mapsto \text{Tr} A \ln A - \text{Tr} A \ln B\).

There are alternative proofs, for instance, by **Epstein** and **Effros**.

There is also a proof of SSA using the non-commutative version of Minkowski’s inequality:
Minkowski’s Inequality for Traces

**Theorem. (Carlen–Lieb (1999 and 2008))** For $1 \leq q \leq p \leq 2$, and all positive operators $A$ on $\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \mathcal{H}_3$,

\[
\text{Tr}_3 \left( \text{Tr}_2 \left[ \left( \text{Tr}_1 A^q \right)^{p/q} \right] \right)^{q/p} \leq \text{Tr}_3 \left( \text{Tr}_1 \left[ \left( \text{Tr}_2 A^p \right)^{q/p} \right] \right)
\]  

For $0 \leq p \leq 1$, and any $q \geq p$, this inequality reverses.

Since (1) is an equality at $p = q = 1$, we can obtain inequalities by differentiating in $p$ at $p = q = 1$. Since

\[
\frac{d}{dp} \text{Tr}(\rho^p) \bigg|_{p=1} = \text{Tr}(\rho \ln \rho),
\]

the resulting inequality will involve the entropies of various partial traces of $\rho$. In fact, we obtain the strong subadditivity of the quantum entropy (SSA):

\[
S(\rho_{13}) + S(\rho_{23}) \geq S(\rho_{123}) + S(\rho_3).
\]

The Minkowski inequality must have other uses for entropy inequalities. Find them!
Another way to enter the circle

The **Triple Matrix Inequality** *(Lieb, 1973)* states that for operators $X, Y, Z > 0$,

$$
\text{Tr} \ e^{\ln X - \ln Y + \ln Z} \leq \int_0^\infty dt \ \text{Tr} \ X(Y + t)^{-1}Z(Y + t)^{-1}
$$

(The ordinary Golden–Thompson inequality is recovered if one of $X, Y, Z$ is the identity.)

The original proof is somewhat complicated, but it is the only proof so far! Find a better one.

Let’s see how this implies **SSA**. By the **Gibbs** variational principle,

$$
S(\rho_{12}) + S(\rho_{23}) - S(\rho_{123}) - S(\rho_2) = \text{Tr}_{123} \ H \rho_{123} - S(\rho_{123}) \geq -\ln \text{Tr}_{123} \ e^{-H},
$$

with the ‘Hamiltonian’ $H = -\ln \rho_{12} + \ln \rho_2 - \ln \rho_{23}$. Thus, by **Triple Matrix Ineq.**,

$$
\text{Tr}_{123} \ e^{-H} \leq \int_0^\infty dt \ \text{Tr}_{123} \rho_{12}(\rho_2 + t)^{-1}\rho_{23}(\rho_2 + t)^{-1}
$$

$$
= \int_0^\infty dt \ \text{Tr}_2 \rho_2(\rho_2 + t)^{-1}\rho_2(\rho_2 + t)^{-1} = \text{Tr}_2 \rho_2 = 1. \quad \square
$$
**Lower bound for SSA**

Recall the inequality $S(\rho_{12}) + S(\rho_{23}) \geq S(\rho_1) + S(\rho_3)$ which is equivalent to SSA and which is deduced from it by 'purification'.

Add to this the equally valid inequality $S(\rho_{13}) + S(\rho_{23}) \geq S(\rho_1) + S(\rho_2)$ and obtain

$$S(\rho_{12}) + S(\rho_{13}) + 2S(\rho_{23}) \geq 2S(\rho_1) + S(\rho_2) + S(\rho_3)$$

which, by purification, is equivalent to

$$S(\rho_{13}) + S(\rho_{34}) - S(\rho_{134}) - S(\rho_3) \geq 2(S(\rho_1) - S(\rho_{14})).$$

By symmetry, we obtain, finally (Carlen–Lieb, 2012)

$$S(\rho_{13}) + S(\rho_{34}) - S(\rho_{134}) - S(\rho_3) \geq 2 \max \{S(\rho_1) - S(\rho_{14}), S(\rho_4) - S(\rho_{14}), 0\}.$$

Classically, $S(\rho_1) - S(\rho_{14}) \leq 0$, so this says, that when we are in the **quantum regime** ($S(\rho_1) - S(\rho_{14}) > 0$ or $S(\rho_4) - S(\rho_{14}) > 0$) we have a bound on SSA.

(Christandl–Winter (2004) had this earlier with the average $\frac{1}{2}(S(\rho_1) + S(\rho_4)) - S(\rho_{14})$, which can be very different.)
AN ASIDE: ENTROPY AND UNCERTAINTY

Here is a recent application of **Triple matrix** (Frank–Lieb 2013). If $\sum_j A_j^* A_j = \sum_k B_k^* B_k = 1$ on $\mathcal{H}_1$, then for any density matrix $\rho_{12}$ on $\mathcal{H}_1 \otimes \mathcal{H}_2$,

$$H(1^A\vert 2) + H(1^B\vert 2) \geq S(1\vert 2) - 2 \ln c_1$$

where $c_1 := \sup_{j,k} \sqrt{\text{Tr}_1 B_k^* A_j^* A_j B_k^*}$, $S(1\vert 2) := S(\rho_{12}) - S(\rho_2)$ and

$$H(1^A\vert 2) := -\sum_j \text{Tr}_2 (\text{Tr}_1 A_j \rho_{12} A_j^*) \ln (\text{Tr}_1 A_j \rho_{12} A_j^*) - S(\rho_2)$$

Under additional rank-one assumptions on $A_j$ and $B_k$ this is due to Berta et al., Coles et al., Tomamichel–Renner, 2010–2012. For trival $\mathcal{H}_2$, this is due to Maassen–Uffink. Our theorem is also valid for general POVMs (for instance Fourier transform).

To prove this, use **operator Jensen** (not needed for rank one) to bound

$$H(1^A\vert 2) + H(1^B\vert 2) - S(1\vert 2) \geq \text{Tr}_{12} H \rho_{12} - S(\rho_{12})$$

with ‘Hamiltonian’ $H = -\ln \sum_j A_j^* A_j (\text{Tr}_1 A_j \rho_{12} A_j^*) - \ln \sum_k B_k^* B_k (\text{Tr}_1 B_k \rho_{12} B_k^*) + \ln \rho_2$. Now apply **Gibbs** and **Triple matrix exactly** as in the proof of **SSA**.
**Next topic: Other notions of entropy**

**Rényi entropy:**
\[(\alpha - 1)^{-1} \ln \text{Tr} \rho^\alpha \sigma^{1-\alpha}\]

This is **monotone under CPTP maps** if \(\alpha \in [0, 2]\). (Follows from Lieb convexity/Ando concavity plus the argument (CONV) \(\implies\) (MONO) from before.)

**Sandwiched Rényi entropy:** (Wilde et al., Müller-Lennert et al.)
\[(\alpha - 1)^{-1} \ln \text{Tr} \left( \sigma^{\frac{1-\alpha}{2\alpha}} \rho \sigma^{\frac{1-\alpha}{2\alpha}} \right)^\alpha\]

**Frank–Lieb 2013** showed that this is **monotone under CPTP maps** if \(\alpha \in [1/2, \infty)\). (Later alternative proofs in certain parameter regimes.) Our proof uses again Lieb convexity/Ando concavity plus (CONV) \(\implies\) (MONO), but also another ingredient, to be discussed below.

**Even more general:** \(\alpha - z\) **entropies:** Jaksic et al., Audenaert–Datta
\[D_{\alpha,z}(\rho||\sigma) := (\alpha - 1)^{-1} \ln \text{Tr} \left( \sigma^{\frac{1-\alpha}{2z}} \rho^{\frac{\alpha}{z}} \sigma^{\frac{1-\alpha}{2z}} \right)^z\]

**Open problem:** For which values of \(\alpha\) and \(z\) is this **monotone under CPTP maps**?
The Audenaert–Datta Conjecture

Conjecture: (Audenaert–Datta, 2015) Monotonicity under CPTP maps holds for
\[ D_{\alpha,z}(\rho||\sigma) = (\alpha - 1)^{-1} \ln \text{Tr} \left( \sigma^{1-\alpha} \rho^{\frac{\alpha}{z}} \sigma^{\frac{1-\alpha}{z}} \right)^z \]
iff \( 0 < \alpha < 1 \), \( z \geq \max\{\alpha, 1 - \alpha\} \) or \( 1 < \alpha < \infty \), \( \max\{\alpha/2, \alpha - 1\} \leq z \leq \alpha \).

Hiai (2013) has shown that these conditions are necessary for monotonicity and has shown monotonicity for \( 0 < \alpha < 1 \). (\( 0 < \alpha < 1 \) and \( z = 1 \) is Lieb concavity.)
Monotonicity holds for \( 1 < \alpha \leq 2 \) and \( z = 1 \) by Ando convexity. Moreover, by Frank–Lieb (2013) monotonicity holds for \( 1 < \alpha < \infty \) and \( z = \alpha \).
Carlen–Frank–Lieb (2014) have shown monotonicity for \( 1 < \alpha \leq 2 \) and \( z = \alpha/2 \).
Thus, for \( \alpha > 1 \) the conjecture has been proved on all the ‘endpoint’ lines, but there remains a lot of work to be done...

All monotonicity results come from convexity/concavity results for
\[ (\rho, \sigma) \mapsto \text{Tr} \left( \sigma^{1-\alpha} \rho^{\frac{\alpha}{z}} \sigma^{\frac{1-\alpha}{z}} \right)^z. \]
This leads us to...
NEW CONVEXITY/CONCAVITY THEOREMS

The Audenaert–Datta conjecture stimulated work on convexity/concavity properties of more general trace functionals

\[(A, B) \mapsto \text{Tr} \left( B^{q/2} A^p B^{q/2} \right)^s \] (*)

**Theorem 1 (Concavity).** Let \( p, q \in \mathbb{R} \) and \( s > 0 \). Then (*) is concave iff \( 0 \leq p, q \leq 1 \) and \( s \leq 1/(p + q) \).

**Theorem 2 (Convexity).** If

\[ p \in \{1, 2\}, \ -1 \leq q \leq 0 \text{ and } s \geq 1/(p + q) \]
or

\[ p \in (1, 2), \ -1 \leq q \leq 0 \text{ and } s \geq \min\{1/(p - 1), 1/(q + 1)\}, \]

then (*) is convex.

**Remarks.**

1. Most of Thm. 1 is due to Hiai. Carlen–Frank–Lieb removed his restriction \( s \geq \frac{1}{2} \).
2. Thm. 2 is due to Carlen–Frank–Lieb. From Hiai we know that for \( p \in [1, 2] \), the condition \( s \geq 1/(p + q) \) is necessary, so for \( p \in \{1, 2\} \) we have an ‘iff’ result. For \( p \in (1, 2) \) convexity is conjectured also for \( 1/(p + q) \leq s < \min\{1/(p - 1), 1/(q + 1)\} \).
3. Convexity is conjectured for \(-1 \leq p, q < 0 \) and \( 0 < s < 1/2 \).
A SAMPLE PROOF

Claim. \((A, B) \mapsto \text{Tr} \left( B^{\frac{q}{2}} A^p B^{\frac{q}{2}} \right)^s \) is convex for \( p \in [1, 2] \), \( q \in (-1, 0) \), \( s \geq 1/(1 + q) \).

Proof. Following Carlen–Lieb (2008) we use the `quasi-linearization formula'

\[
\text{Tr} \left( B^{\frac{q}{2}} A^p B^{\frac{q}{2}} \right)^s = \sup_{Z \geq 0} \left( s \text{Tr} B^{\frac{q}{2}} A^p B^{\frac{q}{2}} Z - (s - 1) \text{Tr} Z^{\frac{s}{s-1}} \right).
\]

Now change variables \( D^2 = B^{\frac{q}{2}} Z B^{\frac{q}{2}} \) to get

\[
\text{Tr} \left( B^{\frac{q}{2}} A^p B^{\frac{q}{2}} \right)^s = \sup_{D \geq 0} \left( s \text{Tr} D A^p D - (s - 1) \text{Tr} (DB^{-q} D)^{\frac{s}{s-1}} \right).
\]

Since a supremum of convex functions is convex, it suffices to prove that (1) \( A \mapsto \text{Tr} D A^p D \) is convex and (2) \( B \mapsto \text{Tr} (DB^{-q} D)^{\frac{s}{s-1}} \) is concave, for each fixed \( D \geq 0 \).

Note that we have decoupled \( A \) and \( B \)!

(1) follows since \( A \mapsto A^p \) is operator convex for \( 1 \leq p \leq 2 \).

(2) follows from Hiai's extension of theorems of Epstein and Carlen–Lieb. (Here \(-qs/(s-1) \leq 1\), that is, \( s \geq 1/(1 + q) \), is used.) \(\square\)
**Final topic: Joint operator convexity**

For the proof of convexity for $s \geq 1/(p - 1)$ we need to study operator convexity/concavity properties of

$$(A, B) \mapsto B^{\frac{q}{2}} A^p B^{\frac{q}{2}} \quad (**)$$

**Lemma 3.** Let $p, q \in \mathbb{R} \setminus \{0\}$. Then

- $(**)$ is convex iff $-1 \leq p < 0$ and $q = 2$
- $(**)$ is not concave

This lemma is closely connected to triple convexity/concavity properties of

$$(A, B, C) \mapsto \text{Tr} C^{\frac{r}{2}} B^{\frac{q}{2}} A^p B^{\frac{q}{2}} C^{\frac{r}{2}} \quad (***)$$

**Lemma 4.** Let $p, q, r \in \mathbb{R} \setminus \{0\}$. Then

- $(***)$ is convex iff $q = 2, p, r < 0$ and $p + r \geq -1$
- $(***)$ is not concave

The positive results are in *Lieb* (’73) and the negative results in *Carlen–Frank–Lieb* (’14). These lemmas are somewhat disappointing, but they show how subtle matrix inequalities are!
THANK YOU FOR YOUR ATTENTION!