

Strengthened Monotonicity of Relative Entropy via Pinched Petz Recovery Map

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Result (overview)

- New proof of [Fawzi and Renner, arXiv:1410.0664] based on concavity and monotonicity of the operator logarithm
- Strengthening/generalization of the inequality

Notation

- *Relative entropy*: $D(\rho\|\sigma) := \text{tr}(\rho(\log \rho - \log \sigma))$ if $\text{supp}(\rho) \subseteq \text{supp}(\sigma)$ and $+\infty$ otherwise
- (von Neumann) *entropy*: $H(\rho) := -\text{tr}(\rho \log \rho)$
- *Conditional mutual information (CMI)*:
 $I(A : C|B) := H(AB) + H(BC) - H(ABC) - H(B)$
- *Fidelity*: $F(\rho, \sigma) := \|\sqrt{\rho}\sqrt{\sigma}\|_1$
- Measured relative entropy $D_{\mathbb{M}}(\rho\|\sigma) := \sup\{D(\mathcal{M}(\rho)\|\mathcal{M}(\sigma)) : \mathcal{M}(\rho) = \sum_x \text{tr}(\rho M_x)|x\rangle\langle x| \text{ with } \sum_x M_x = \text{id}\}$, where $\{|x\rangle\}$ is a finite set of orthonormal vectors

What is known

- 1973 Strong subadditivity: $I(A : C|B) \geq 0$ [3, 4]
- 1975 Monotonicity of relative entropy (data processing inequality):
 $D(\rho\|\sigma) - D(\mathcal{N}(\rho)\|\mathcal{N}(\sigma)) \geq 0$ [5, 8]
- Oct. 2014 For any ρ_{ABC} there exists a recovery map $\mathcal{R}_{B \rightarrow BC}$ such that [2]
 $I(A : C|B) \geq -2 \log F(\rho_{ABC}, \mathcal{R}_{B \rightarrow BC}(\rho_{AB}))$
Recovery map has the form of a rotated Petz recovery map
- Nov. 2014 For any ρ_{ABC} there exists a recovery map $\mathcal{R}_{B \rightarrow BC}$ such that [1]
 $I(A : C|B) \geq D_{\mathbb{M}}(\rho_{ABC}\|\mathcal{R}_{B \rightarrow BC}(\rho_{AB}))$
 $\geq -2 \log F(\rho_{ABC}, \mathcal{R}_{B \rightarrow BC}(\rho_{AB}))$
- April 2015 For CMI lower bound there exists a *universal* recovery map (that only depends on ρ_{BC}) & unitaries commute [7]
- May 2015 For any σ, \mathcal{N} there exists a recovery map $\mathcal{R}_{\sigma, \mathcal{N}}$ (rotated Petz with commuting unitaries) such that $(\mathcal{R}_{\sigma, \mathcal{N}} \circ \mathcal{N})(\sigma) = \sigma$ and [9]
 $D(\rho\|\sigma) - D(\mathcal{N}(\rho)\|\mathcal{N}(\sigma)) \geq -2 \log F(\rho, (\mathcal{R}_{\sigma, \mathcal{N}} \circ \mathcal{N})(\rho))$

Pinched and rotated Petz recovery map

- For $H = \sum_x \lambda_x |x\rangle\langle x|$ let $P_\lambda := \sum_{x:\lambda_x=\lambda} |x\rangle\langle x|$ and define the *pinching map*

$$\mathcal{P}_H : \mathbb{P}(A) \ni X \mapsto \sum_{\lambda \in \text{spec}(H)} P_\lambda X P_\lambda \in \mathbb{P}(A)$$

- *Pinching recovery map* for $n \in \mathbb{N}$

$$\mathcal{R}_{\sigma, \mathcal{N}}^n : \mathbb{P}(B^{\otimes n}) \rightarrow \mathbb{P}(A^{\otimes n})$$

$$X_{B^n} \mapsto (\sigma^{\frac{1}{2}})^{\otimes n} \mathcal{P}_{\sigma^{\otimes n}} \left((\mathcal{N}^\dagger)^{\otimes n} \left[(\mathcal{N}(\sigma)^{-\frac{1}{2}})^{\otimes n} \mathcal{P}_{\mathcal{N}(\sigma)^{\otimes n}}(X_{B^n}) (\mathcal{N}(\sigma)^{-\frac{1}{2}})^{\otimes n} \right] (\sigma^{\frac{1}{2}})^{\otimes n} \right)$$

- For $\sigma = \sum_{k \in [d_1]} \lambda_k P_k$ let $U_\sigma^\vartheta := \sum_{k \in [d_1]} \exp(i\vartheta_k) P_k$ with $\vartheta \in [0, 2\pi]^{d_1}$ let us define a *rotated Petz recovery map*

$$\mathcal{T}_{\sigma, \mathcal{N}}^{\vartheta, \vartheta} : \mathbb{P}(B) \rightarrow \mathbb{P}(A)$$

$$X_B \mapsto U_\sigma^\vartheta \sigma^{\frac{1}{2}} \mathcal{N}^\dagger(\mathcal{N}(\sigma)^{-\frac{1}{2}} U_\sigma^\vartheta X_B U_\sigma^{\vartheta\dagger} \mathcal{N}(\sigma)^{-\frac{1}{2}}) \sigma^{\frac{1}{2}} U_\sigma^{\vartheta\dagger}$$

- For any $n \in \mathbb{N}$

$$\mathcal{R}_{\sigma, \mathcal{N}}^n(\cdot) = \frac{1}{(2\pi)^{d_1}} \int_{[0, 2\pi]^{d_1}} d\vartheta \frac{1}{(2\pi)^{d_2}} \int_{[0, 2\pi]^{d_2}} d\varphi (\mathcal{T}_{\sigma, \mathcal{N}}^{\vartheta, \vartheta})^{\otimes n}(\cdot) \quad (1)$$

Result (formal)

Let

$$\mathbb{T}_{\sigma, \mathcal{N}} := \text{conv} \left(\mathcal{T}_{\sigma, \mathcal{N}}^{\vartheta, \vartheta} : \vartheta \in [0, 2\pi]^{d_1}, \varphi \in [0, 2\pi]^{d_2} \right)$$

Main result

For any $\sigma \in \mathbb{P}(A)$, any $\rho \in \mathbb{S}_\sigma(A)$, and any $\mathcal{N} \in \text{TPCP}(A, B)$ there exists a recovery map $\mathcal{R}_{\sigma, \mathcal{N}, \rho} \in \mathbb{T}_{\sigma, \mathcal{N}}$, such that

$$D(\rho\|\sigma) - D(\mathcal{N}(\rho)\|\mathcal{N}(\sigma)) \geq D_{\mathbb{M}}(\rho\|(\mathcal{R}_{\sigma, \mathcal{N}, \rho} \circ \mathcal{N})(\rho))$$

$$\stackrel{\text{easy}}{\geq} -2 \log F(\rho, (\mathcal{R}_{\sigma, \mathcal{N}, \rho} \circ \mathcal{N})(\rho)) .$$

- The recovery map satisfies $(\mathcal{R}_{\sigma, \mathcal{N}, \rho} \circ \mathcal{N})(\sigma) = \sigma$
- For $\rho = \rho_{ABC}$, $\sigma = \rho_{BC}$, and $\mathcal{N}(\cdot) = \text{tr}_C(\cdot)$ we reproduce [2, 1]
- The second inequality was proved in [9] using Hadamard's three line theorem

Proof

Proposition $D(\rho\|\sigma) - D(\mathcal{N}(\rho)\|\mathcal{N}(\sigma)) \geq \liminf_{n \rightarrow \infty} \frac{1}{n} D(\rho^{\otimes n}\|(\mathcal{R}_{\sigma, \mathcal{N}}^n \circ \mathcal{N}^{\otimes n})(\rho^{\otimes n}))$

Proof sketch for the Proposition.

- $\mathcal{P}_H(X)$ commutes with H
- Pinching inequality: $\mathcal{P}_H(X) \geq \frac{1}{|\text{spec}(H)|} X$ for all $X \in \mathbb{P}(A)$
- For any $\rho \in \mathbb{P}(A)$ we have $|\text{spec}(\rho^{\otimes n})| = O(\text{poly}(n))$
- Operator logarithm is concave and monotone

□

With the proposition the main result follows from (1) together with the "Piani" argument [6], which shows that for any $n \in \mathbb{N}$

$$\frac{1}{n} \min_{\mu \in \mathbb{P}(\mathbb{T}_{\sigma, \mathcal{N}})} D(\rho^{\otimes n}\|\int d\mu(\vartheta, \varphi) (\mathcal{T}_{\sigma, \mathcal{N}}^{\vartheta, \vartheta} \circ \mathcal{N})(\rho)^{\otimes n})$$

$$\geq \min_{\mu \in \mathbb{P}(\mathbb{T}_{\sigma, \mathcal{N}})} D_{\mathbb{M}}(\rho\|\int d\mu(\vartheta, \varphi) (\mathcal{T}_{\sigma, \mathcal{N}}^{\vartheta, \vartheta} \circ \mathcal{N})(\rho))$$

Discussion and open problems

- Universality: a recovery map that does not depend on ρ [to appear] (Here the measured relative entropy bound becomes important)
- Does the Petz recovery map satisfy all these inequalities?
- The pinching recovery map (for $n = 1$) does not satisfy the inequalities

References

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