

Approximate degradable quantum channels

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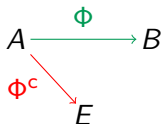
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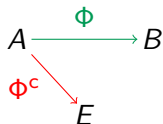
ETH zürich

Quantum capacity of a channel

- ▶ Quantum channel (TPCPM) $\Phi : \mathcal{S}(A) \rightarrow \mathcal{S}(B)$
 - ▶ By Stinespring $\Phi : \rho_A \mapsto \text{tr}_E(V_{BE}\rho_A V_{BE}^\dagger)$
 - ▶ Complementary channel $\Phi^c : \rho_A \mapsto \text{tr}_B(V_{BE}\rho_A V_{BE}^\dagger)$

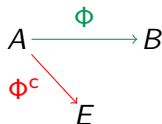


Quantum capacity of a channel



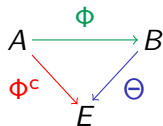
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- ▶ How much (quantum) information can we reliably send over such a channel?
 - ▶ Quantum capacity [Lloyd-Shor-Devetak-97]
 $Q(\Phi) = \lim_{k \rightarrow \infty} \frac{1}{k} Q^{(1)}(\Phi^{\otimes k})$
 - ▶ Coherent information
 $Q^{(1)}(\Phi) := \max_{\rho \in \mathcal{S}(A)} H(\Phi(\rho)) - H(\Phi^c(\rho))$ with
 $H(\rho) := -\text{tr}(\rho \log \rho)$

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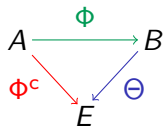
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- ▶ “Problems” with the LSD-formula
 - ▶ Regularization makes it difficult to compute
 - ▶ $Q^{(1)}(\Phi) \leq Q(\Phi)$ however $Q^{(1)}(\Phi) < Q(\Phi)$ possible [DiVincenzo-Shor-Smolin-98]
 - ▶ Single letter upper bounds are difficult to find
 - ▶ Would like to have UBs that are efficiently computable

Degradable channels



- ▶ A channel $\Phi : \mathcal{S}(A) \rightarrow \mathcal{S}(B)$ is *degradable* if \exists a channel $\Theta : \mathcal{S}(B) \rightarrow \mathcal{S}(E)$ such that $\Phi^c = \Theta \circ \Phi$.
- ▶ If Φ is degradable then $Q^{(1)}(\Phi) = Q(\Phi)$ [Devetak-Shor-05]

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- ▶ If Φ is degradable then $Q^{(1)}(\Phi) = Q(\Phi)$ [Devetak-Shor-05]
- ▶ Examples of degradable channels
 - ▶ Dephasing channels, e.g. $\rho \mapsto (1 - p)\rho + pX\rho X$
 - ▶ Amplitude damping channels
- ▶ Not all channels are degradable ☹
 - ▶ Depolarizing channel, i.e., $\rho \mapsto (1 - p)\rho + p\pi$.
 - ▶ BB84 channel (independent bit and phase flip error)
- ▶ Concept of degradable channels is not robust

Approximate degradable channels

$$\text{TPCPM } \Xi : \mathcal{S}(A) \rightarrow \mathcal{S}(B)$$

$$\|\Xi\|_{\diamond} := \max_{\rho \in \mathcal{S}(A \otimes A')} \|(\Xi \otimes \mathcal{I}_{A'}) (\rho)\|_1$$

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- ▶ A channel $\Phi : \mathcal{S}(A) \rightarrow \mathcal{S}(B)$ is ε -*degradable* if \exists a channel $\Theta : \mathcal{S}(B) \rightarrow \mathcal{S}(E)$ such that $\|\Phi^c - \Theta \circ \Phi\|_{\diamond} \leq \varepsilon$.
- ▶ Every channel is ε -degradable with some $\varepsilon \in [0, 2]$

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Theorem. Let Φ be ε -degradable, then

$$Q^{(1)}(\Phi) \leq Q(\Phi) \leq Q^{(1)}(\Phi) + 6\varepsilon \log(n_E) + 4h(\varepsilon),$$

with $n_E := \dim E$ and $h(x) := -x \log x - (1-x) \log(1-x)$

A few remarks about how to prove the theorem

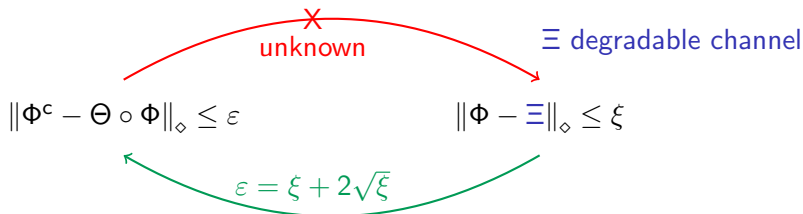
- ▶ Alicki-Fannes inequality: If $\|\rho_{AB} - \sigma_{AB}\|_1 \leq \varepsilon$ then $|H(A|B)_\rho - H(A|B)_\sigma| \leq 4\varepsilon \log(n_A) + 2h(\varepsilon)$
- ▶ Following the Devetak-Shor proof and applying Alicki-Fannes a few times
- ▶ Degradability is used via the data processing inequality, i.e., $I(A : B) \geq I(A : E)$

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An important comment

Unclear if ε -degradable channels are close to a degradable channel.
Channels that are close to degradable ones are ε -degradable



Approximate degradable channels (con't)

- ▶ A channel $\Phi : \mathcal{S}(A) \rightarrow \mathcal{S}(B)$ is ε -degradable if \exists a channel $\Theta : \mathcal{S}(B) \rightarrow \mathcal{S}(E)$ such that $\|\Phi^c - \Theta \circ \Phi\|_{\diamond} \leq \varepsilon$
- ▶ How to find the smallest ε such that Φ is ε -degradable?

$$\varepsilon_{\Phi} := \begin{cases} \min_{\Theta} & \|\Phi^c - \Theta \circ \Phi\|_{\diamond} \\ \text{s. t.} & \Theta : \mathcal{S}(B) \rightarrow \mathcal{S}(E) \text{ is tpcp} \end{cases} \quad (1)$$

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Proposition. (1) can be expressed as a semidefinite program

- ▶ $Q^{(1)}(\Phi) \leq Q(\Phi) \leq Q^{(1)}(\Phi) + 6\varepsilon_{\Phi} \log(n_E) + 4h(\varepsilon_{\Phi})$ is efficiently computable

Proof sketch of the proposition

- ▶ The diamond norm of a difference of two channels can be phrased as an SDP [Watrous-09]

$$\|\Xi_1 - \Xi_2\|_\diamond = \begin{cases} \inf_Z & \|\text{tr}_B(Z)\|_\infty \\ \text{s. t.} & Z \geq J(\Xi_1 - \Xi_2) \\ & Z \geq 0 \end{cases}$$

Choi state of $\Xi_1 - \Xi_2$

- ▶ The mapping $J(\Theta) \mapsto J(\Theta \circ \Phi)$ is linear, thus

$$\begin{aligned} \varepsilon_\Phi &= \begin{cases} \inf_\Theta & \|\Phi^c - \Theta \circ \Phi\|_\diamond \\ \text{s. t.} & \Theta : \mathcal{S}(\mathcal{H}_B) \rightarrow \mathcal{S}(\mathcal{H}_E) \text{ is tpcp} \end{cases} \\ &= \begin{cases} \inf_{Z, J(\Theta)} & \|\text{tr}_E(Z)\|_\infty \\ \text{s. t.} & Z \geq J(\Phi^c) - J(\Theta \circ \Phi) \\ & Z \geq 0 \\ & J(\Theta) \geq 0 \\ & \text{tr}_E(J(\Theta)) = \mathbb{1}_B. \end{cases} \end{aligned}$$

Approximate concavity of the coherent information

- ▶ Coherent information $I_c(\rho, \Phi) := H(\Phi(\rho)) - H(\Phi^c(\rho))$
- ▶ $Q^{(1)}(\Phi) := \max_{\rho \in \mathcal{S}(\mathcal{H}_A)} I_c(\rho, \Phi)$

Proposition. Let $\Phi : \mathcal{S}(\mathcal{H}_A) \rightarrow \mathcal{S}(\mathcal{H}_B)$ be a channel that is ε -degradable with $n_E := \dim \mathcal{H}_E$. Then,

$$I_c\left(\sum_x p(x)\rho_x, \Phi\right) \geq \sum_x p(x)I_c(\rho_x, \Phi) - 6\varepsilon \log(n_E) - 4h(\varepsilon),$$

where p is a probability distribution and $\rho_x \in \mathcal{S}(\mathcal{H}_A)$.

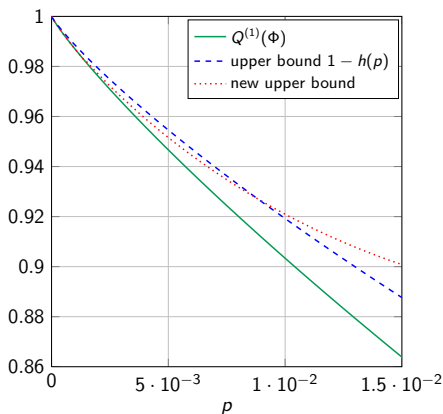
- ▶ Simplifies the task of computing $Q^{(1)}(\Phi)$

First application: depolarizing channel

$$\Phi : \rho \mapsto (1 - p)\rho + p\mathbb{1}, \text{ for } p \in [0, 1]$$

Universal hashing bound

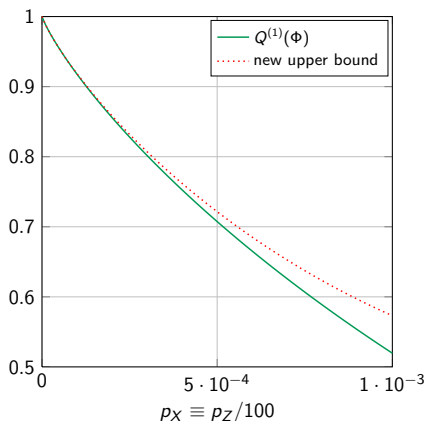
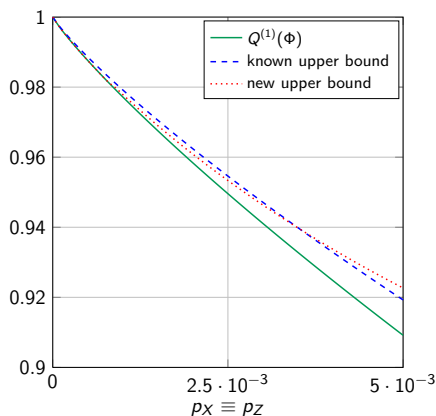
$$Q^{(1)}(\Phi) = 1 + (1 - p) \log(1 - p) + p \log\left(\frac{p}{3}\right)$$



Second application: BB84 channel

Independent bit and phase error $\Phi : \rho \mapsto (1 - p_X - p_Z + p_X p_Z)\rho + (p_X - p_X p_Z)X\rho X + (p_Z - p_Z p_X)Z\rho Z + p_X p_Z Y\rho Y$

$$Q^{(1)}(\Phi) = 1 - h(p_X) - h(p_Z)$$



Comments to existing upper bounds

- ▶ Convex decomposition into degradable channels
[Smith-Smolín-Winter-08]
 - ▶ $\Phi = \sum_i p_i \Theta_i$, where $\{\Theta_i\}_i$ are degradable
 - ▶ $Q(\sum_i p_i \Theta_i) \leq \sum_i p_i Q(\Theta_i) = \sum_i p_i Q^{(1)}(\Theta_i)$
 - ▶ Channel specific ☹
 - ▶ Decomposition into degradable channels may not exist!
- ▶ The quantum capacity with symmetric side channels
[Smith-Smolín-Winter-08]
- ▶ No cloning argument [Cerf & Bruss *et al.*-98]
 - ▶ Only good at very high noise levels
- ▶ New approach offers
 - ▶ **universal** upper bound (method works for *any* channel)
 - ▶ UB is **efficiently computable** (via an SDP)
 - ▶ UB is **good at low noise levels** (ideal channel is degradable)

What about high noise levels?

- ▶ A channel $\Phi : \mathcal{S}(A) \rightarrow \mathcal{S}(B)$ is *anti-degradable* if \exists a channel $\Theta : \mathcal{S}(E) \rightarrow \mathcal{S}(B)$ such that $\Phi = \Theta \circ \Phi^c$.
- ▶ Anti-degradable channels cannot have positive quantum capacity (no-cloning)
- ▶ A channel $\Phi : \mathcal{S}(A) \rightarrow \mathcal{S}(B)$ is *ε -anti-degradable* if \exists a channel $\Theta : \mathcal{S}(E) \rightarrow \mathcal{S}(B)$ such that $\|\Phi - \Theta \circ \Phi^c\|_{\diamond} \leq \varepsilon$.

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Proposition. If Φ is ε -anti-degradable, then $Q(\Phi) \leq 6\varepsilon \log n_B + 4h(\varepsilon)$.

- ▶ Proof works similar as for the ε -degradable case

Another implication of the theorem

Theorem. Let Φ be ε -degradable, then

$$Q^{(1)}(\Phi) \leq Q(\Phi) \leq Q^{(1)}(\Phi) + 6\varepsilon \log(n_E) + 4h(\varepsilon)$$

- ▶ Every channel is ε -degradable with some $\varepsilon \in [0, 2]$

Thus for any channel Φ , we have $Q(\Phi) - Q^{(1)}(\Phi) \leq 12 \log(n_E)$

- ▶ Independent of n_B
- ▶ Constants probably can be improved

Private classical capacity of a quantum channel

Private classical capacity of Φ

$$P(\Phi) = \lim_{k \rightarrow \infty} \frac{1}{k} P^{(1)}(\Phi^{\otimes k}),$$

with

$$P^{(1)}(\Phi) := \max_{\{\rho_i, p_i\}} H\left(\sum_i p_i \Phi(\rho_i)\right) - \sum_i p_i H(\Phi(\rho_i)) \\ - H\left(\sum_i p_i \Phi^c(\rho_i)\right) + \sum_i p_i H(\Phi^c(\rho_i))$$

- ▶ $P^{(1)}(\Phi) \leq P(\Phi)$ and $P^{(1)}(\Phi) < P(\Phi)$ possible
[Smith-Renes-Smolin-08]
- ▶ For degradable channels $P^{(1)}(\Phi) = P(\Phi) = Q^{(1)}(\Phi) = Q(\Phi)$
[Smith-08]

Private classical capacity of a quantum channel (con't)

- ▶ For degradable channels $P^{(1)}(\Phi) = P(\Phi) = Q^{(1)}(\Phi) = Q(\Phi)$

Theorem. Let Φ be ε -degradable, then

$$P^{(1)}(\Phi) \leq P(\Phi) \leq P^{(1)}(\Phi) + 14\varepsilon \log(n_E) + 8h(\varepsilon)$$

and

$$Q^{(1)}(\Phi) \leq P^{(1)}(\Phi) \leq Q^{(1)}(\Phi) + 6\varepsilon \log(n_E) + 4h(\varepsilon)$$

- ▶ Efficient computable upper bounds for $P(\Phi)$

Summary & outlook

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- ▶ Robust definition of degradable channels
- ▶ Approximately preserve properties of degradable channels
 - ▶ additivity of coherent information
- ▶ Useful for upper bounds to the quantum capacity
 - ▶ computable via SDP
- ▶ Same for private classical capacity of a quantum channel
- ▶ Useful to prove upper bounds for the quantum capacity of bosonic channels? (see open problem from yesterday!)