Workshop Report:
Groups, Graphs and Stochastic Processes

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1 Overview

We had a busy week, with 42 participants, 22 talks, and a lively open problem session. The workshop was successful in bringing together experts and young researchers from the related fields of asymptotic group theory, ergodic theory, L^2 invariants, geometric group theory, random walks, statistical physics and algebraic graph theory.

We had a number of superb expositions of new and upcoming developments in the field. Seven of the talks were by current graduate students and post-docs. A brief description of talks is given below, as are the open problems presented. Highlights of the week include Bordenave's work on the spectral gap of $G(n,p)$, Chatterjee's work related to Yang-Mills theory in dimension $N \to \infty$, and Lyons' talk about his proof with Peres of the Kaimanovich-Vershik conjecture.

2 Summary of talks

2.1 Monday

Charles Bordenave opened the workshop with his new proof of Friedman's eigenvalue theorem. The proof is a significant simplification compared to previous arguments, and extends to lifts of other graphs as well.

Matthias Keller followed and described the theory behind the Royden compactification of a graph: a boundary defined in terms of bounded functions of finite Dirichlet energy.

Sourav Chatterjee gave an exciting talk with a very accessible introduction to his recent work around the Yang-Mills problem for lattice gauge theories.

Perla Sousi discussed the set of points unvisited by a random walk in a torus in dimension $d \geq 3$. At time αt_{cov}, where t_{cov} is the expected cover time, the set is expected to have size $n^{(1-\alpha)d}$. It is close in distribution to a uniform subset of the torus if α is large enough, but far from a uniform set if α is small. The sharpness of the phase transition is not known:

Problem 2.1. Is there some α_c so that the total variation distance tends to 1 if $\alpha < \alpha_c$ and to 0 if $\alpha > \alpha_c$?

Ander Holroyd closed the first day with a remarkable talk about Finitely dependent coloring, describing processes he recently discovered with Liggett which have a number of unusual and unexpected properties.

Problem 2.2. Is the described 2-dependent 3-colouring of $\mathbb{Z}$ a factor of i.i.d.? The 4-colouring is a factor of i.i.d.; what the possible tail of the factor radius?

2.2 Tueday

Mustazzee Rahman described his work with Virág on the scaling limits of random sorting networks. They show that the conjectured scaling limit of the uniform sorting network is the unique minimizer under the Wasserstein-2 metric on measures on the square. He also discussed large deviations for the interchange process, explaining why sorting networks are likely to minimize that length.

Problem 2.3 ([**?**]). Show that paths in a large uniform sorting network are with high probability close to sine-curves.

James Lee showed us a program to study mixing times of Markov chains using *modified* log-Sobolev inequalities of the form $\frac{d}{dt}H(P^t f) \leq -c\mathcal{E}(f, \log f)$. When this holds, we get exponential decay of the entropy of $P^t f$, which implies mixing. On some chains (for example S_n with random transpositions) this method can give better bounds than those derived from log-Sobolev inequalities.

Russ Lyons gave a proof (with Yuval Peres) of the Kaimanovich-Vershik conjecture, that the Poisson boundary of the lamp-lighter group on $\mathbb{Z}^d$ is given by the eventual lamp configuration. The elegant proof is based on estimating the entropy of the walk conditioned on the eventual configuration. The proof extends to some more general random walks and some other graphs. However, some settings are still open.

Problem 2.4. Does the Kaimanovich-Vershik conjecture hold for random walks with steps of infinite entropy? What about lamp-lighter on arbitrary transient sub-graphs of $\mathbb{Z}^d$?

Nicolas Matte Bon explained the notion of Extensively amenable actions, a strengthening of amenability which can be used to prove certain groups are amenable by considering their actions on carefully chosen spaces. Extensive amenability also has a nice probabilistic interpretation. With his co-authors [**?**], they can show that certain subgroups of the interval exchange transformation (IET) group are amenable, and give a criterion for amenability of IET and of Thompson's group.

Problem 2.5. If the Schreier graph of a group G acting on X has uniformly polynomial volume growth, is the action extensively amenable?

Ron Peled talked about permutations weighted by $q^{inv(\sigma)}$. The typical size and variance of cycles is estimated. It is also shown that the longest increasing subsequence in such permutations has length of order $n\sqrt{1-q}$ for q sufficiently close to 1 [**?**, **?**].

Problem 2.6. Do the longest cycle lengths, suitably normalized converge in law? When is the limit Poisson-Dirichlet?

2.3 Wednesday

Peter Csikvari described works Abért, Frenkel, Hubai and Kun on the notion of the *matching measure*, related to the entropy of matchings of a specified size on a graph. They prove among other things that these quantities are continuous under the local topology.

David Gamarnik described his work with Li and Sudan, on upper bounds on effectiveness of local algorithms for k-SAT and MAX-CUT approximations. They build on the methods of Gamarnik and Sudan, and of Mustazzee and Virág who proved similar upper bounds on independent sets in random graphs.

Gabor Kun solved Bowen's problem to prove that local approximations of a finitely generated group with Kazhdan Property (T) are essentially expanders, in the sense that they can be separated into disjoint expanders with $o(n)$ edges between them.

2.4 Thursday

Lewis Bowen gave a clear introduction to the notion of *F-entropy*, a form of entropy for invariant processes on free groups, extending also to other groups. This overcomes some obstacles in using entropy on non-amenable groups, and skirts others.

Agnes Backhausz talked of her work with Bálint Virág concerning invariant Gaussian processes on regular trees. They give criteria for when such a process is a factor of i.i.d. in terms of absolute continuity w.r.t. the i.i.d. process and in terms of the spectral measures of the processes.

Andrew Stewart proved, also with Virág, that the trace of random walks on the free group conditioned to return to the origin converges to the CRT. This extends work of Baugerol and Jeulin who essentially proved that for the simple random walk. The proof is based on a self-similarity characterization of the CRT due to Aldous.

Ori Parzanchevski discussed work with Rozenthal about high-dimensional expanders, where simplicial complexes replace graphs. It turns out that (even beyond expansion), the topology of the complex has probabilistic consequences for random walks. Interestingly, some generalizations of the Alon-Boppana spectral gap bound fail in this setting.

Yuval Peres gave a talk about random walks on the giant component of $G(n,p)$ and $G(n,d)$. While the mixing time from some vertices is of order $\log^2 n$, from most vertices the walk mixes in $O(\log n)$ steps. With Berestycki, Lubetzky and Sly, they also show cutoff for mixing from a typical vertex, and bound the mixing window. There is an interesting distinction between regular and irregular graphs: in the former the mixing occurs at the same time that the walk reaches the distance to typical vertices. In the latter, Some additional time is needed, since the harmonic measure on an irregular tree is far from uniform.

Tianyi Zheng gave an overview of her exciting work with Saloff-Coste, bounding the entropy of a random walk on a group in terms of the decay of the return probabilities. While exponential decay implies non-amenability and linear entropy, they show that stretched exponential decay implies a corresponding polynomial bound on the entropy.

Doron Puder gave the last talk of the day, on his works with Hall and Sawin constructing Ramanujan k-covers of graphs. The extend the breakthrough work of Marcus, Spielman and Srivastava [?] who were able to find Ramanujan 2-covers.

2.5 Friday

We finished the meeting with two talks on Friday.

Tom Hutchcroft described his work with Nachmias, proving indistinguishability of trees in uniform spanning forests, that is that any invariant property of a tree either holds for all or for none of the components. The proof is based on ideas of Lyons and Schramm (in the context of percolation), with a form of update tolerance replacing insertion tolerance, though some cases need a very different treatment.

Problem 2.7. When is the free uniform spanning forest connected? It is shown that it has either 1 or infinitely many components.

Soumik Pal closed the workshop with his work with Ganguly about doubly evolving random graphs. The configuration model on $2d$-regular graphs can evolve both in time, with edge rewiring dynamics, and in size, where a vertex is removed and edges of its neighbours rewired. This gives a two parameter process of random graphs. General linear statistics of the spectral measures of the graph are known to be Gaussian, and the resulting Gaussian processes are studied.

3 Open Problems

As part of the workshop we held an open problem session. We solicited problems which people find interesting and have thought about, but that are not well known, and which can be stated in a few minutes without an extensive background. We thank Lior Silberman for a draft of this list.

3.1 Problem from Doron Puder

Let Γ be a graph on n vertices, G a (finite) group, $\pi \in \hat{G}$ a d-dimensional representation. For each oriented edge $\vec{e} \in E(\Gamma)$ independently choose a random element $g_e \in G$ (with the inverse element assigned to the reverse direction). We obtain a random $(nd) \times (nd)$ matrix in $M_n\left(M_d(\mathbb{C})\right)$ by putting for each edge present the representation matrix $\pi(g_{\vec{e}})$.

Problem 3.1. Is there a choice making the resulting matrix "Ramanujan"?

It is shown by Hall, Puder and Sawin [?] that the answer is "Yes" provided that $\left\{\bigwedge^i \pi\right\}_{i=1}^{d}$ are irreducible and non-isomorphic and the representation behaves well under complex conjugation. In particular, this is the case for $G = S_d$ and π the standard (permutation modulu constants) representation.

3.2 Problem from Miklós Abért

The *local-global* topology on graphs is defined as follows. A sequence of finite graphs G_n converge if for every $r, k \geq 1$ and $\varepsilon > 0$ there is an N such that if $m, n > N$, for every colouring $c : G_n \to [k]$ there is a colouring $c' : G_m \to [k]$ such that the law of coloured balls of radius r around a uniform vertex in G_n and in G_m are within ε total-variation distance of each other (see Bollobás and Riordan [**?**]). This topology is easily metrizable, and extends the weak local topology (Benjamini-Schramm) which is the case $k = 1$. However, the local-global topology is far less understood. In particular, the following is open.

Conjecture 3.2. Let $G(n, d)$ be uniform random d-regular graphs on n points. Do they converge in the local-global sense?

A corollary would be that the independence ratio, max-cut ratio and others would converge. Some of these are known by results of Gamarnik, Tetali and XXX [**?**].

3.3 Problem from Ron Peled

Consider the grid $\Lambda_n = [n] \times [n]$, and consider the set L_n of homomorphisms from Λ_n to G, (i.e. all $f : \Lambda_n \to \mathbb{Z}$ such that $|f(z) - f(w)| = 1$ when z, w are neighbours), pinned by $f(u) = 0$ for a corner u. Choose $f \in L_n$ uniformly at random. Let v be the opposite corner.

Problem 3.3. Prove that $\mathrm{Var}(f(v)) \asymp \log n$.

The best known upper bound is $O(n)$.

Problem 3.4. Consider instead 1-Lipschitz functions $f : \Lambda_n \to \mathbb{R}$, pinned at u with the natural uniform measure. Prove that $\mathrm{Var}(f(v)) \asymp \log n$. Here the lower bound is known.

Question 3.5. Is there a formula for the number of maps? **Answer**: the exponential growth rate is known.

Question 3.6. Is the problem easier in high dimension? **Answer**: The variance is conjectured to be $O(1)$. For high enough dimension this is known (Peled [**?**]).

3.4 Problem from Ryokichi Tanaka

Let $\mathbb{D}$ be the Poincaré disc, Γ a surface group with standard generating set S, and embed $\mathrm{Cay}(\Gamma; S)$ in $\mathbb{D}$ by an orbit. Consider the harmonic measure on the boundary $\partial\Gamma = \partial\mathbb{D}$ of a simple random walk on Γ.

Problem 3.7. Is the harmonic measure absolutely continuous w.r.t. Lebesgue measure, or is it singular to it?

This is known if Γ non-uniform, but open if Γ is uniform = cocompact.

On the other hand, consider the Gromov boundary $\partial_G \,\mathrm{Cay}\,(\Gamma; S)$. This is again homeomophic to S^1(bi-Hölder but not bi-Lipschitz), but (if Γ is not virtually free) the harmonic measure is known to be singular with respect to Hausdorff measure.

Remark (Ecaterina Sava-Huss)**.** Julio Tiozzo has recent work on this.

3.5 Problem from Bálint Virág

Let Γ be a Cayley graph, $W : \Gamma \to \mathbb{R}$ an invariant Gaussian process with $\mathbb{E}W_e^2 < \infty$. Let X_n be SRW on Γ started at e. The **spectral radius** is defined by

$$\rho_W = \lim_{n\to\infty} \left|\mathrm{Cov}\left(W_e, W_{X_{2n}}\right)\right|^{1/2n}.$$

Also define

$$\rho_{\mathrm{proc}}(\Gamma) = \sup\Big(\{\rho_W \mid W \text{a process}\} \setminus \{1\}\Big).$$

Finally, let ρ_Γ be the spectral radius of Γ.

Theorem 3.8. *$\rho_\Gamma \le \rho_{proc}(\Gamma) \le 1$ and $\rho_{proc}(\Gamma) < 1$ if and only if Γ has the Kazhdan (T) property.*

Note also that the spectral radius of the RW on Γ has $\rho_\Gamma = 1$ iff Γ is amenable, and so for amenable groups $\rho_{\mathrm{proc}}(\Gamma) = 1$.

Example 3.9. If Γ is free, $\rho_\Gamma < 1$ but $\rho_{\mathrm{proc}}(\Gamma) = 1$.

Problem 3.10. Find a non-amenable Γ such that $\rho_\Gamma = \rho_{\mathrm{proc}}(\Gamma)$.

3.6 Problem from M. Kassabov

Γ finitely generated group with generating set S. B_n ball of radius n in the Cayley graph, $\mu_{\mathrm{RW},n}$ the random walk measure on B_n, $\mu_{C,n}$ the counting measure. Let $\Gamma \twoheadrightarrow Q$ be a finite quotient. When $n \to \infty$, the projections of $\mu_{RW,n}$ are long RW measures on Q, that is converge to the uniform measure.

The projections of $\mu_{C,n}$ can converge but not to the uniform measure.

Conjecture 3.11. $\Gamma = \mathrm{SL}_n(\mathbb{Z})$, $Q = \mathrm{SL}_n(\mathbb{Z}/p\mathbb{Z})$, then for all S the projections converge to the uniform measure.

3.7 Problem from Ecaterina Sava-Huss

(Inspired by talk by Russ Lyons.)

Take the infinite d-regular tree, and fix a root v_0 and an end ξ. Consider the random walk which walks "up" (toward ξ) with probability $\frac{1}{2}$, and "down" uniformly.

Fact 3.12. The height of this random walk is a simple random walk on $\mathbb{Z}$, but nevertheless the walk is transient, and converges to ξ. More precisely (Peres), for any finite subset of the tree, the walk is eventually absorbed in the infinite component of the complement containing ξ.

Consider now the lamplighter group on the tree, where there is a $\{0, 1\}$ lamp at each vertex. The random walk may flip the lamp, or move as above.

Problem 3.13. Is the Poisson boundary for this lamplighter walk here generated by the eventual lamp configuration? Do the techniques of Lyons and Peres (see Lyons's talk) apply?

3.8 Problem from M. Keller

T a tree, $\{\omega_x\}_{x\in V(T)}$ i.i.d. random variables valued in $[0,1]$. Let A be the adjacency matrix and consider the Schrödinger operator

$$H_\omega^\lambda = A + \lambda\omega\,.$$

Let $V_\omega^\lambda = \operatorname{Span}\left\{\left(H_\omega^\lambda\right)^n \delta_0\right\}_{n=0}^{\infty}$.

Problem 3.14. Is there almost surely $v^\omega \in \ell^2(T)$ orthogonal to V_ω^λ.

This would give an easy proof of delocalization (ac spectrum).

Question 3.15. What about the line? **Answer**: this fails there.

3.9 Problem from Ander Holroyd

Problem 3.16. Does there exist a Markov chain $\{X_n\}$ on some compact metric space and some $k, \varepsilon > 0$ such that

1. The chain mixes perfectly in k steps: $\mathbb{P}_x(X_k \in \cdot) = \pi(\cdot)$, and
2. The walk moves at least ε: For every x, $\mathbb{P}_x(d(X_1, x) < \varepsilon) = 0$.

Example 3.17. The walk on $[0,1]^2$ where from (x,y) the chain jumps to (y,U) with uniform U mixes in two steps, but may stay arbitrarily close to some points.

This cannot be done with a finite state space. There is an example with a countable, non-compact state space.

Question 3.18 (Virág)**.** Do you know that this impossible for S^1? **Answer**: No.

3.10 Problem from David Gamarnik

Let $G = (V,E)$ be an infinite graph with maximum degree $\Delta < \infty$. To each $v \in V$ we would like to associate some $X_v \in [0,1]$ subject to linear constraints: for each edge $(u,v) \in E$ we impose

$$\alpha_{u,v} X_u + \beta_{u,v} X_v \le \gamma_{u,v}\,.$$

(Even the case $\alpha, \beta, \gamma \equiv 1$ is interesting.)

Let $\{X_v\}$ be chosen uniformly with respect to the product Lebesgue measure subject to the constraints (assuming the set of solutions has positive measure).

Problem 3.19. In the ball of radius r, place boundary conditions on the the sphere, and consider the induced conditional distribution on the value at the center. Is it true that, in the limit of $r \to \infty$ the marginal on the center is independent of the boundary conditions?

Conjecture 3.20. For every graph, for every reasonable set of constraints (at least including $X_u + X_v \le 1$) there is asymptotic independence.

Remark.

1. This includes a variant of Peled's question.
2. If $X_u \in \{0,1\}$ this becomes the hard core model, where there is no uniqueness.
3. For the regular tree the claim holds (Gamarnik–Ramanan [**?**]).
4. (Peres): non-uniqueness is compatible with solvability.

3.11 Problems from Yuval Peres

Random walk on amenable Cayley graphs. Let Γ be an amenable Cayley graph of exponential growth (the non-amenable case is known). Let X_n be the simple random walk on Γ. We know the return probability decays sub-exponentially.

Problem 3.21. Is it true that the entropy satisfies $H(X_n) \geq c\sqrt{n}$?

The best known lower bound is $\Omega(n^{1/3})$ (Rokhlis: all transition probabilities are bounded above by $e^{-n^{1/3}}$).

Can also consider return probability to the typical point:

Problem 3.22. Is it true that with high probability $P^n(X_0, X_n) \leq e^{-cn^{1/2}}$?

Both of these bounds are sharp for the lamplighter on $\mathbb{Z}$.

Occupation times of balls. Let Γ be a transient Cayley graph. How long can you remain in the group of radius r?

Problem 3.23. In any group, is it true that $\sum_n \mathbb{P}(X_n \in B_r) \leq Cr^2$?

It is known and non-trivial that the *exit time* is at most quadratic. (unpublished proof by Erschler, extended and unpublished by Virág, described in print by Lee–Peres [**?**]).