On the combinatorial structure of Arthur packets: p-adic symplectic and orthogonal groups

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Global motivation

- $k$ number field, $\mathbb{A}$ adele ring of $k$
- $\Gamma_k$ absolute Galois group, $W_k$ Weil group
- $G$ split symplectic or special odd orthogonal group over $k$
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Definition

Automorphic representations of $G(\mathbb{A})$ are irreducible constituents of the regular representation on $L^2(G(k) \backslash G(\mathbb{A})).$

\[
L^2(G(k) \backslash G(\mathbb{A})) = L^2_{disc}(G) \oplus L^2_{cont}(G)
\]

\[
L^2_{disc}(G) = L^2_{cusp}(G) \oplus L^2_{res}(G)
\]
Global Langlands Correspondence

\[
\left\{ \text{discrete automorphic representations of } G(\mathbb{A}) \right\} / \sim \leftrightarrow \left\{ \text{discrete Arthur parameters } \psi : L_k \times SL(2, \mathbb{C}) \to \hat{G} \right\} / \hat{G} - \text{conj}
\]
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\]

$L_k$ is the hypothetical global Langlands group satisfying

\[
1 \longrightarrow K_k \longrightarrow L_k \longrightarrow W_k \longrightarrow 1
\]

where $K_k$ is compact.
Arthur packet

- $\mathcal{A}_2(G)$: equivalence classes of discrete automorphic representations.
- $\Psi_2(G)$: equivalence classes of discrete Arthur parameters.
Arthur packet

- $\mathcal{A}_2(G)$: equivalence classes of discrete automorphic representations.
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**Theorem (Arthur)**

For each $\psi \in \Psi_2(G)$, there exits a “multi-set” $\Pi_\psi$ of equivalence classes of irreducible admissible representations of $G(\mathbb{A})$ such that

1. $\Pi_\psi = \bigotimes'_v \Pi_{\psi_v}$

2. $\mathcal{A}_2(G) \subseteq \bigsqcup_{\psi \in \Psi_2(G)} \Pi_\psi$

3. *(Endoscopy theory)*: One can distinguish the automorphic representations in $\Pi_\psi$. 
Arthur packet

- $\Pi_\psi$ is called **global Arthur packet**
- $\Pi_{\psi_v}$ is a finite “multi-set” of equivalence classes of irreducible admissible representations of $G(k_v)$, called **local Arthur packet**.
Arthur packet

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**Example**

1. $G = \text{SO}(3) \cong \text{PGL}(2)$: $\Pi_\psi$ is a single automorphic representation.

2. $G = \text{Sp}(2) \cong \text{SL}(2)$: $\Pi_{\psi_v}$ is the restriction of an automorphic representation of $\text{GL}_2(\mathbb{A})$ to $\text{SL}_2(\mathbb{A})$. 
Global problem

How to distinguish the residue spectrum in $\Pi_\psi$?
Global problem

How to distinguish the *residue spectrum* in $\Pi_\psi$?

Mœglin:
- global condition: zeros (poles) of certain L-functions “related” to $\psi$.
- local condition: “fine” parametrization of $\Pi_{\psi_v}$. 
Let $F$ be a $p$-adic field, $L_F = W_F \times SL(2, \mathbb{C})$

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Let $\hat{G} \xrightarrow{std.} GL_N(\mathbb{C})$ ($N = 2n$ or $2n+1$) be the standard representation.

$\psi : W_F \times SL(2, \mathbb{C}) \times SL(2, \mathbb{C}) \rightarrow \hat{G} \xrightarrow{std.} GL(N, \mathbb{C})$

with bounded image on $\psi|_{W_F}$. 

Arthur parameter
Jordan blocks

\[ \psi = \bigoplus_i (\rho_i \otimes \nu_{a_i} \otimes \nu_{b_i}) \]

- \( \rho_i \) equivalence class of unitary irreducible representation of \( W_F \)
- \( a_i, b_i \in \mathbb{N} \)
- \( \nu_m \) is \( \text{Sym}^{m-1} \)-representation of \( SL(2, \mathbb{C}) \)
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Define

\[ \text{Jord}(\psi) = \{(\rho_i, a_i, b_i)\} \]

and

\[ \text{Jord}_\rho(\psi) := \{(\rho', a', b') \in \text{Jord}(\psi) : \rho' = \rho\}. \]
Parity

For self-dual $\rho$: orthogonal type or symplectic type

\[(\rho, a, b) \text{ is orthogonal} \iff \begin{cases} a + b \text{ is even, if } \rho \text{ is orthogonal} \\ a + b \text{ is odd, if } \rho \text{ is symplectic} \end{cases}\]

\[(\rho, a, b) \text{ is symplectic} \iff \begin{cases} a + b \text{ is odd, if } \rho \text{ is orthogonal} \\ a + b \text{ is even, if } \rho \text{ is symplectic} \end{cases}\]
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$$\psi_p = \bigoplus_{(\rho, a, b) \in \text{Jord}(\psi)} \rho \otimes \nu_a \otimes \nu_b$$

same parity as $\hat{G}$

From now on, we will assume $\psi = \psi_p$. 
Visualize Jordan blocks

For \((\rho, a, b) \in \text{Jord}(\psi)\),

\[
A = \frac{(a + b)}{2} - 1 \quad B = \frac{|a - b|}{2}
\]

and

\[
\zeta = \begin{cases} 
\text{Sign}(a - b), & \text{if } a \neq b \\
\text{arbitrary}, & \text{otherwise.}
\end{cases}
\]

So we can also denote \((\rho, a, b)\) by \((\rho, A, B, \zeta)\).
**Admissible order**

A total order $\succ_{\psi}$ on $\text{Jord}_{\rho}(\psi)$ is called admissible if

$$\forall (\rho, A, B, \zeta), (\rho, A', B', \zeta') \in \text{Jord}_{\rho}(\psi) \text{ satisfying }$$

$$A \succ A', B \succ B' \text{ and } \zeta = \zeta'$$

we have $(\rho, A, B, \zeta) \succ_{\psi} (\rho, A', B', \zeta').$

**Example**
**Discrete diagonal restriction**

**Definition**
We say $\psi$ has **discrete diagonal restriction** if for each $\rho$ the Jordan blocks in $\text{Jord}_\rho(\psi)$ are “disjoint”.

![Diagram showing the concept of discrete diagonal restriction with labeled points and symbols representing the blocks and their relationships.](image-url)
Discrete diagonal restriction

Definition

We say $\psi$ has discrete diagonal restriction if for each $\rho$ the Jordan blocks in $Jord_\rho(\psi)$ are “disjoint”.

In this case, $Jord_\rho(\psi)$ has a natural order $>_{\psi}$, namely

$$(\rho, A, B, \zeta) >_{\psi} (\rho, A', B', \zeta')$$

if and only if $A > A'$. 
Mœglin’s parametrization I

Theorem (Mœglin)
Suppose $\psi$ has discrete diagonal restriction, $\succ \psi$ is the natural order,

$$\Pi_\psi = \bigoplus \frac{\pi_{M, \succ \psi} (\psi, l, \eta)}{\{(l, \eta): \Pi_{(\rho, a, b) \in \text{Jord}(\psi)} \in_l, \eta (\rho, a, b) = 1 \} / \sim}$$

where $\pi_{M, \succ \psi} (\psi, l, \eta)$ is irreducible.
Mœglin’s parametrization I

**Theorem (Mœglin)**

Suppose \( \psi \) has discrete diagonal restriction, \( \succ \psi \) is the natural order,

\[
\Pi_\psi = \bigoplus \{ (l, \eta) : \Pi_{(\rho, a, b) \in \text{Jord}(\psi)} \varepsilon_{l, \eta}(\rho, a, b) = 1 \} / \sim 
\]

where \( \pi_{M, \succ \psi}(\psi, l, \eta) \) is irreducible.

- \((l, \eta)\) are integral valued functions over \( \text{Jord}(\psi) \), such that
  \[l(\rho, A, B, \zeta) \in [0, [(A - B + 1)/2]] \text{ and } \eta(\rho, A, B, \zeta) \in \{\pm 1\},\]

- \(\varepsilon_{l, \eta}(\rho, A, B, \zeta) := \eta(\rho, A, B, \zeta)^{A-B+1}(-1)^{(A-B+1)/2+l(\rho, A, B, \zeta)}\)

- \((l, \eta) \sim (l', \eta')\) if and only if \(l = l'\), and
  \[\left(\frac{\eta}{\eta'}\right)(\rho, A, B, \zeta) = 1\]

unless \(l(\rho, A, B, \zeta) = (A - B + 1)/2\).
Dominating parameter

For $\psi$ and admissible $\triangleright_{\psi}$, we can index $Jord_{\rho}(\psi)$ such that

$$(\rho, A_i, B_i, \zeta_i) \triangleright_{\psi} (\rho, A_{i-1}, B_{i-1}, \zeta_{i-1}).$$

We say $\psi \gg \psi$ dominates $\psi$ with respect to $\triangleright_{\psi}$ if $Jord_{\rho}(\psi \gg \psi)$ consists of

$$(\rho, A_i + T_i, B_i + T_i, \zeta_i)$$

for $T_i \geq 0$,

with the same admissible order $\triangleright_{\psi}$ under the natural identification.
Mœglin’s parametrization II

For $\psi$ and admissible $>\psi$, we choose a dominating parameter $\psi\gg$ with discrete diagonal restriction. Then we define

$$\pi_{M,>\psi}(\psi, I, \eta) := \circ_{(\rho, A_i, B_i, \zeta_i) \in \text{Jord}_\rho(\psi) \text{Jac}_i} \pi_{M,>\psi}(\psi\gg, I, \eta).$$
Mœglin’s parametrization II

For \( \psi \) and admissible \( \succ \psi \), we choose a dominating parameter \( \psi \succ \) with discrete diagonal restriction. Then we define

\[
\pi_{M, \succ \psi}(\psi, l, \eta) := \circ \rho; (\rho, A_i, B_i, \xi_i) \in \text{Jord}_\rho(\psi) \text{Jac}_X_i \pi_{M, \succ \psi}(\psi \succ, L, \eta).
\]

**Proposition (Mœglin)**

1. \( \pi_{M, \succ \psi}(\psi, l, \eta) \) is either irreducible or zero.
2. If \( \pi_{M, \succ \psi}(\psi, l, \eta) = \pi_{M, \succ \psi}(\psi, l', \eta') \neq 0 \), then \( (l, \eta) \sim (l', \eta') \).
3. \[
\Pi_\psi = \bigoplus_{\{(l, \eta) : \text{Jac}_X \in \text{Jord}(\psi) \in \xi \eta, (\rho, a, b) = 1\}} \pi_{M, \succ \psi}(\psi, l, \eta).
\]

where \( \pi_{M, \succ \psi}(\psi, l, \eta) \) is irreducible or zero.
Local problem

What are the conditions on \((l, \eta)\) for \(\pi_{M, \psi}(\psi, l, \eta) \neq 0\) ?

Example
Pull

1.
Pull

2

3
Change sign

$A_n + B_n$
Example

Let $\psi = \rho \otimes \nu_{51} \otimes \nu_{31} \oplus \rho \otimes \nu_{31} \otimes \nu_{45} \oplus \rho \otimes \nu_{13} \otimes \nu_{5}$. Then

$[A_3, B_3] = [40, 10] \quad [A_2, B_2] = [37, 7] \quad [A_1, B_1] = [8, 4]$
Example

\[ 0 \leq l_1 \leq 2, 0 \leq l_2 \leq 15, 0 \leq l_3 \leq 15, \text{ and } (-1)^{l_1 + l_2 + l_3} \eta_1 \eta_2 \eta_3 = 1. \]
Example

\[ 0 \leq l_1 \leq 2, 0 \leq l_2 \leq 15, 0 \leq l_3 \leq 15, \text{ and } (-1)^{l_1 + l_2 + l_3} \eta_1 \eta_2 \eta_3 = 1. \]

| \eta_3 = \eta_1 \text{ and } \eta_2 = \eta_1 | -5 \leq l_3 - l_2 + 2l_1 \leq 15 |
| \eta_3 = \eta_1 \text{ and } \eta_2 \neq \eta_1 | l_3 + l_2 + 2l_1 > 25 |
| \eta_3 \neq \eta_1 \text{ and } \eta_2 = \eta_1 | l_3 - l_1 < 11 + l_1 \text{ and } l_3 + l_2 - 2l_1 > 15 |
| \eta_3 \neq \eta_1 \text{ and } \eta_2 = \eta_1 | l_3 - l_1 \geq 11 + l_1 \text{ and } -36 \leq -l_3 - l_2 + 2l_1 \leq -16 |
| \eta_3 \neq \eta_1 \text{ and } \eta_2 \neq \eta_1 | l_3 - l_1 < 11 + l_1 \text{ and } -15 \leq l_3 - l_2 - 2l_1 \leq 5 |
| \eta_3 \neq \eta_1 \text{ and } \eta_2 \neq \eta_1 | l_3 - l_1 \geq 11 + l_1 \text{ and } -l_3 + l_2 + 2l_1 > -6 |

Each case gives rise to a polytope, and by counting the integral points in them we get \(|\Pi_\psi| = 1651|\).