Generating weights for modules of vector-valued modular forms

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Motivations

- Partition functions of modules over VOA $\rightarrow$ vector-valued modular forms.

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- Partition functions of modules over VOA $\rightarrow$ vector-valued modular forms.
- Theory of general vector-valued modular forms for integral weight $k \in \mathbb{Z}$ (Bantay, Gannon, Knopp, Marks, Mason,...)
- From number theory: vectors of theta functions, Weil representation, Jacobi forms, mostly weight $\frac{1}{2}\mathbb{Z}$ (Borcherds, Bruinier, Eichler-Zagier, Skoruppa, ...)

Main Problem

Let $\rho : \mathrm{SL}_2(\mathbb{Z}) \to \mathrm{GL}(V)$ be a complex f.d. representation, $M_k(\rho)$ the space of holomorphic vector-valued modular forms of weight k,

$$M(\rho) := \bigoplus_{k \in \mathbb{Z}} M_k(\rho),$$

viewed as a graded module over $M(1) = \mathbb{C}[E_4, E_6]$.

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Vector valued modular forms

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Definition

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$$f(\gamma\tau) = (c\tau + d)^k \rho(\gamma)\, f(\tau)$$

for all $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathsf{SL}_2(\mathbb{Z})$.

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- ρ gives a local system $\mathcal{V}(\rho)$ over $\mathscr{M}$.
- f is a section of $\mathcal{V}_k(\rho) := \mathcal{V}(\rho) \otimes \mathcal{L}_k$.

Geography of $\overline{\mathscr{M}}$

The compactification $\overline{\mathscr{M}}$ of $\mathscr{M}$ is given by the diagram:

$$\begin{tikzcd}
& {[\langle -I_2, T\rangle \backslash \mathfrak{h}]} \arrow[dl, "\iota_1"'] \arrow[r, leftrightarrow, "\tau \mapsto e^{2\pi i \tau} = q"'] & {[\mu_2 \backslash \mathbf{D}^\times]} \arrow[dr, "\iota_2"] & \\
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Vector bundles on $\overline{\mathscr{M}}$ correspond to triples $(\mathcal{V}, \mathcal{W}, \phi)$, where $\mathcal{V}$ is a vector bundle on $\mathscr{M}$, $\mathcal{W}$ is a vector bundle on $[\mu_2\backslash\mathbf{D}]$ and

$$\phi : \iota_1^*\mathcal{V} \simeq \iota_2^*\mathcal{W}$$

is a bundle isomorphism lying over $\tau \mapsto q$.

Examples

(1) $\overline{\mathcal{L}}_k := (\mathcal{L}_k, \mathcal{M}_k, \phi(z,\tau) = (z,q))$, where $\mathcal{M}_k = [\mu_2 \backslash \mathbb{C} \times \mathbf{D}]$, action given by $(\pm 1)(z,q) = ((\pm 1)^k z, q)$.

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Suppose all eigenvalues of L have real part in $(0, 1]$. Then $H^0(\overline{\mathscr{M}}, \overline{\mathcal{V}}_{L,k}(\rho) =: \overline{\mathcal{S}}_k(\rho)) = S_k(\rho)$, the space of ρ-valued cusp forms of weight k.

The free-module Theorem

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Theorem (Mason-Marks, Gannon, C.-Franc,...)

(i) *$M(\rho)$ is a free module of rank $n = \dim\rho$ over $M(1)$.*

(ii) *If $k_1 \leq \ldots \leq k_n$, $k_j \in \mathbb{Z}$, are the weights of the free generators, then*

$$\sum_{j=1}^{n} k_j = 12\,\mathrm{Tr}(L).$$

(iii) *If ρ is unitary, then $0 \leq k_j \leq 11$.*

Finding the generating weights

Main Question

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- We have

$$\sum_{k \in \mathbb{Z}} \dim M_k(\rho) t^k = \frac{t^{k_1} + \ldots + t^{k_n}}{(1-t^4)(1-t^6)} \in \mathbb{Z}[\![t]\!]$$

so the question is equivalent to finding $\dim M_k(\rho)$ for all k.

The metaplectic group

- The metaplectic group $\mathrm{Mp}_2(\mathbb{Z})$ is the unique nontrivial central extension

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- Multiplication:
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- Generators: $T := \left(\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right), 1\right), \quad S := \left(\left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right), \sqrt{\tau}\right)$

Vector valued modular forms of weight $k \in \frac{1}{2}\mathbb{Z}$

Let $\rho : \mathrm{Mp}_2(\mathbb{Z}) \to \mathrm{GL}(V)$ be complex, finite-dimensional representation.

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- Growth conditions at ∞ are specified by a matrix L such that $\rho(T) = e^{2\pi i L}$
- Denote by $M_k(\rho)$ (resp. $S_k(\rho)$) the space of holomorphic modular forms (resp. cusp forms).

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Theorem (C., Franc, 2015)

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so the question is equivalent to finding $\dim M_k(\rho)$ for all k.

Finite quadratic modules

Definition

A finite quadratic module is a pair (D, q) of a finite abelian group D together with a quadratic form $q : D \to \mathbb{Q}/\mathbb{Z}$, whose associated bilinear form we denote by $b(x, y) := q(x + y) - q(x) - q(y)$.

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E.g.

For $m > 0$ even, let $\Lambda = (\mathbb{Z}, x \mapsto \frac{m}{2}x^2)$, a rank 1 lattice. The discriminant form of Λ is the finite quadratic module

$$A_m := (\mathbb{Z}/m\mathbb{Z}, x \mapsto \frac{x^2}{2m})$$

The Weil Representation

Let (D, q) be a finite quadratic module. Let $\mathbb{C}(D)$ be the $\mathbb{C}$-vector space of functions $f : D \to \mathbb{C}$. This space has a canonical basis $\{\delta_x\}_{x \in D}$ of delta functions, i.e. $\delta_x(y) = \delta_{x,y}$.

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Definition

The Weil representation $\rho_D : \mathrm{Mp}_2(\mathbb{Z}) \longrightarrow \mathrm{GL}(\mathbb{C}(D))$ is defined with respect to the basis $\{\delta_x\}_{x \in D}$ by

$$\rho_D(T)(\delta_x) = e^{-2\pi i\, q(x)} \delta_x$$

$$\rho_D(S)(\delta_x) = \frac{\sqrt{i}^{\mathrm{sig}(D)}}{\sqrt{|D|}} \sum_{y \in D} e^{2\pi i\, b(x,y)} \delta_y,$$

where $\frac{1}{\sqrt{|D|}} \sum_{x \in D} e^{2\pi i q(x)} = \sqrt{i}^{\mathrm{sig}(D)}$.

Generating weights of Weil representations

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E.g.

For $D = A_m$, $k \in \frac{1}{2} + \mathbb{Z}$, we have

$$M_k(\rho_{A_m}) \simeq J_{k+1/2,m/2},$$

i.e. Jacobi forms of weight $k + 1/2$, index $m/2$ and

$$M(\rho_D) \simeq J_{m/2}$$

the (free) $\mathbb{C}[E_4, E_6]$-module of Jacobi forms of index $m/2$.

Attempt to compute $\dim M_k(\rho_D)$ via Riemann-Roch

- Form the vector bundle $\overline{\mathcal{W}}_k(\rho_D)$ over $\overline{\mathscr{M}_{1/2}} = \overline{[\mathrm{Mp}_2(\mathbb{Z})\backslash\mathfrak{h}]}$.

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Theorem (Case A_{2p}, $p > 3$ prime, $k \in 1/2 + \mathbb{Z}$)

Let L_p such that $e^{2\pi i L_p} = \rho_{A_{2p}}(T)$, and with eigenvalues in $[0,1)$.

$$\chi(\overline{\mathcal{W}}_k(\rho_{A_{2p}})) = \frac{5+k}{12}p - \frac{1}{2}\operatorname{Tr}(L_p) + (-1)^{2k}(\delta + \frac{5+k}{12}) + \epsilon_\pm$$

Here

$$\delta := \frac{1}{8}\left(2 + \left(\frac{-1}{p}\right)\right), \quad \epsilon_\pm := \frac{1}{6}\left(1 \pm \left(\frac{p}{3}\right)\right)$$

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(v) $\dim M_{3/2}(\rho) = \chi(\overline{\mathcal{W}}_{3/2}(\rho))$ (i.e. $H^1 = 0$, Skoruppa).

Computations

E.g.

For $p = 5$, the generating weights for $M(\rho_{A_{10}})$ are

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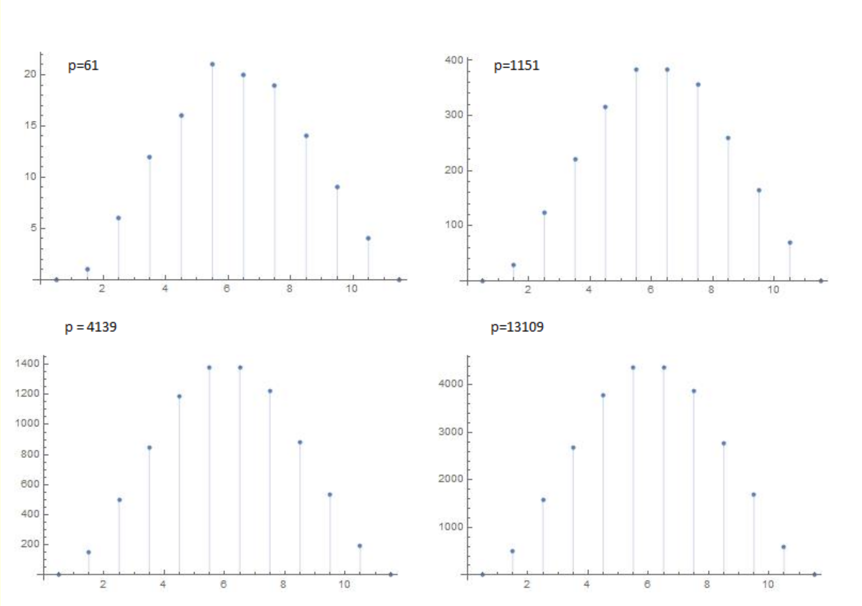
Try some larger primes $p = 61, 1151, 4139, 13109, \ldots$


p=61
p=1151
p = 4139
p=13109

Generating weights for $m = 2p$, $p \geq 5$

weight $= 1/2$	*multiplicity* $= 0$
$3/2$	$\frac{13}{24}(p+1) - \frac{1}{2}\,\mathrm{Tr}(L_p) - \delta - \epsilon_+$
$5/2$	$\frac{15}{24}(p-1) - \frac{1}{2}\,\mathrm{Tr}(L_p) + \delta$
$7/2$	$\frac{17}{24}(p+1) - \frac{1}{2}\,\mathrm{Tr}(L_p) - \delta + \epsilon_+$
$9/2$	$\frac{19}{24}(p-1) - \frac{1}{2}\,\mathrm{Tr}(L_p) + \delta + \epsilon_-$
$11/2$	$\frac{1}{3}(p+1) + \epsilon_+$
$13/2$	$\frac{1}{3}(p-1) - \epsilon_-$
$15/2$	$\frac{-5}{24}(p+1) + \frac{1}{2}\,\mathrm{Tr}(L_p) + \delta - \epsilon_+$
$17/2$	$\frac{-7}{24}(p-1) + \frac{1}{2}\,\mathrm{Tr}(L_p) - \delta - \epsilon_-$
$19/2$	$\frac{-9}{24}(p+1) + \frac{1}{2}\,\mathrm{Tr}(L_p) + \delta$
$21/2$	$\frac{-11}{24}(p-1) + \frac{1}{2}\,\mathrm{Tr}(L_p) - \delta + \epsilon_-$
$23/2$	0

Distribution as $p \to \infty$

Theorem (C., Franc, Kopp, 2016)

Let p be an odd prime and let $m = 2p$, $p > 3$. Then

$$\mathrm{Tr}(L_p) = \begin{cases} p + \frac{1}{2}h_p - \frac{1}{4} & p \equiv 1 \pmod 4, \\ p + 2h_p - \frac{1}{4} & p \equiv 3 \pmod 8, \\ p + h_p - \frac{1}{4} & p \equiv 7 \pmod 8. \end{cases}$$

where h_p is the class number of $\mathbb{Q}(\sqrt{-p})$.

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where h_p is the class number of $\mathbb{Q}(\sqrt{-p})$.

Corollary

If $\rho = \rho_{A_{2p}}$, then

$$\frac{\mathrm{Tr}(L_p)}{2p} \to 1/2, \quad p \to \infty.$$

Distribution of weights for $m = 2p$, as $p \to \infty$

weight = 1/2	*proportion* = 0
3/2	1/48
5/2	3/48
7/2	5/48
9/2	7/48
11/2	8/48
13/2	8/48
15/2	7/48
17/2	5/48
19/2	3/48
21/2	1/48
23/2	0

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Moral of the story

Generating weights might be easier to study than dimensions