Convergence and Holomorphy of Kac–Moody Eisenstein Series

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joint work with L. Carbone, H. Garland, D. Liu and S. D. Miller.

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    - Function field analogue (Braverman, Kazhdan, 2012)

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  - Study of special cases of E_9, E_{10}, E_{11}

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- Today's Answer:

  Yes, if we assume some interesting combinatorial conditions for the Kac–Moody groups.

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  - Extension of the Langlands–Shahidi method to study automorphic L-functions
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  - String theory as will be explained in the talks of Persson and Kleinschmidt

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  W: Weyl group generated by simple reflections w_i, $i \in I$

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- $V_{\mathbb{R}} = \mathbb{R} \otimes_{\mathbb{Z}} V_{\mathbb{Z}}$

  For $s, t \in \mathbb{R}$ and $i \in I$, set

$$\chi_{\alpha_i}(s) = \sum_{n=0}^{\infty} s^n \frac{e_i^n}{n!}, \quad \chi_{-\alpha_i}(t) = \sum_{n=0}^{\infty} t^n \frac{f_i^n}{n!}.$$

  Then $\chi_{\alpha_i}(s)$ and $\chi_{-\alpha_i}(t)$ define elements in $\mathrm{Aut}(V_{\mathbb{R}})$.

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- We have the Iwasawa decomposition

  $$G_{\mathbb{R}} = UA^+K$$

  with uniqueness of expression.

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We define for all $g \in G_{\mathbb{R}}$ the constant term

$$E_\lambda^\sharp(g) = \int_{\Gamma \cap U \backslash U} E_\lambda(ug)du.$$

- Applying the Gindikin–Karpelevich formula, we obtain

$$E^{\sharp}_{\lambda}(g) = \sum_{w \in W} a(g)^{w\lambda+\rho} c(\lambda, w),$$

where

$$c(\lambda, w) = \prod_{\alpha>0, w\alpha<0} \frac{\xi(\langle \lambda, \alpha^{\vee} \rangle)}{\xi(1 + \langle \lambda, \alpha^{\vee} \rangle)},$$

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  $\mathcal{C}^* = \{\lambda \in \mathfrak{h}^* : \langle \lambda, \alpha_i^\vee \rangle > 0, \ i \in I\}$, $\quad \mathfrak{C}^* \subset \mathfrak{h}^*$: dual of $\mathfrak{C}$

  P: weight lattice

Lemma (Looijenga)

Let $\mathcal{K}$ be a compact subset of $\mathfrak{C}$ and $\mu \in P \cap \mathfrak{C}^*$. If $A_{\mathcal{K},\mu}(N)$ is the number of $\mu' \in W \cdot \{\mu\}$ whose maximum on $\mathcal{K}$ is $\geq -N$, then $A_{\mathcal{K},\mu}(N) = O(N^r)$ as $N \to \infty$.

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Theorem

Assume that $\lambda \in \mathfrak{h}^*_{\mathbb{C}}$ with $Re(\lambda) - \rho \in \mathcal{C}^*$. Then $E^\sharp_\lambda(g)$ converges absolutely for $g \in UA_{\mathfrak{C}}K$.

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Theorem

Assume that $\lambda \in \mathfrak{h}_{\mathbb{C}}^*$ with $Re(\lambda) - \rho \in \mathcal{C}^*$. Then $E_\lambda^\sharp(g)$ converges absolutely for $g \in UA_{\mathfrak{C}}K$.

Corollary

For $\lambda \in \mathfrak{h}_{\mathbb{C}}^*$ with $Re(\lambda) - \rho \in \mathcal{C}^*$, there exists a measure zero subset S_0 of $U\mathfrak{S}$ such that the series $E_\lambda(g)$ converges absolutely for $g \in U\mathfrak{S}K$ off the set S_0K, where $\mathfrak{S}$ is an arbitrary compact subset of $A_{\mathfrak{C}}$.

4. Convergence of Eisenstein series

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- Assume that $\lambda - \rho \in \mathcal{C}^*$. Then there exists a constant $M > 0$ depending on λ such that, for $\alpha \in \Delta$, $x \in \mathbb{R}$ and $g = uak \in G_{\mathbb{R}}$,

$$\sum_{m \in \mathbb{Z}} a(w_\alpha u_\alpha(x+m)g)^{\lambda+\rho} \leq M\, a^{w_\alpha(\lambda+\rho)}(1 + a^\alpha),$$

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- Using induction, we want to have, for $w = w_{\beta_1} \dots w_{\beta_\ell}$,

$$(\clubsuit) \quad \sum_{m_1,\dots,m_\ell \in \mathbb{Z}} a(w_{\beta_1}u_{\beta_1}(x_1+m_1)\cdots w_{\beta_\ell}u_{\beta_\ell}(x_\ell+m_\ell)g)^{\lambda+\rho}$$
$$\le M^\ell a^{w^{-1}(\lambda+\rho)} \prod_{\alpha>0, w\alpha<0} (1+a^\alpha).$$

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  Property ($\star$): Assume that $\lambda - \rho \in \mathcal{C}^*$. Every $w \neq \mathrm{id} \in W$ can be written as $w = vw_\beta$ where $\beta \in \Delta$ and $\ell(v) < \ell(w)$, such that for any subset S of $\Phi_+ \cap v^{-1}\Phi_-$ one has

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Theorem

Assume Property ($\star$) and Re(λ) $- \rho \in \mathcal{C}^*$. Then the series $E_\lambda(g)$ converges absolutely for $g \in UA_{\mathfrak{C}}K$.

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Theorem

Assume Property ($\star$) and $Re(\lambda) - \rho \in \mathcal{C}^*$. Then the series $E_\lambda(g)$ converges absolutely for $g \in UA_{\mathfrak{C}}K$.

- Use the inequality (♣) and bound $E_\lambda(g)$ by its constant term.

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- In general, Property ($\star$) is not true. For example, the root system A_3 and w the longest element.
- Property ($\star$) is related to holomorphy of cuspidal Eisenstein series.

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- L: subgroup of $M(\mathbb{R})$ generated by $\chi_{\pm\alpha}(t)$, $\alpha \in \Delta_M$, $t \in \mathbb{R}$

  Then we have $M = LH$.

- Using the Iwasawa decomposition $G_{\mathbb{R}} = NMK$, we define

$$\mathrm{Iw}_L : G_{\mathbb{R}} \to L/L \cap K, \quad \mathrm{Iw}_{H^+} : G_{\mathbb{R}} \to H^+ \cong H/H \cap K.$$

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$$E(s, g) = \sum_{\gamma \in \Gamma \cap P \backslash \Gamma} \mathrm{Iw}_{H^+}(\gamma g)^{s\varpi_P}.$$

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- For an unramified cusp form f on $L(\mathbb{Z})\backslash L(\mathbb{R})$, we define the cuspidal Eisenstein series

$$E_f(s, g) = \sum_{\gamma \in \Gamma \cap P \backslash \Gamma} \mathrm{Iw}_{H^+}(\gamma g)^{s\varpi_P} f\big(\mathrm{Iw}_L(\gamma g)\big).$$

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Definition

A maximal parabolic subgroup $P = MN$ with a finite dimensional Levi subgroup M is said to be *ample* if there exist constants $C, D > 0$ such that for every $w \in W^M$, $w \neq \mathrm{id}$,

(P1) $(C\varpi_P - \rho)(\alpha^\vee) > 0$ for $\alpha \in \Phi'_w$,

(P2) $(D\varpi_P + \rho_M)(\alpha^\vee) < 0$ for $\alpha \in \Phi'_w$,

(P3) $w^{-1}(D\varpi_P + \rho_M)$ is a positive linear combination of simple roots.

Proposition

If P satisfies condition (P1), then for Re $s \geq s_0$ and any compact subset $\mathfrak{S}$ of $A_{\mathfrak{C}}$, there exists a measure zero subset S_0 of $U\mathfrak{S}$ such that $E(s, g)$ converges absolutely for $g \in U\mathfrak{S}K$ off the set S_0K.

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Theorem

If the maximal parabolic subgroup P is ample, then for any compact subset $\mathfrak{S}$ of $A_{\mathfrak{C}}$, there exists a measure zero subset S_0 of $U\mathfrak{S}$ such that $E_f(s,g)$ is an entire function of $s \in \mathbb{C}$ for $g \in U\mathfrak{S}K$ off the set S_0K.

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- We use rapid decay of cusp forms due to Miller and Schmid.

- Let $A_1 = A \cap L$ and $A_1^+ \cong A_1 / A_1 \cap K$.

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- Now we have

$$\begin{aligned}
&\left|\operatorname{Iw}_{H^+}(\gamma g)^{s\varpi_P} f\left(\operatorname{Iw}_L(\gamma g)\right)\right| \\
\leq &C_1 \operatorname{Iw}_{H^+}(\gamma g)^{(\operatorname{Re} s)\varpi_P} \operatorname{Iw}_{A_1^+} \circ \operatorname{Iw}_L(\gamma g)^{-n\rho_M} \\
\leq &C_1 \operatorname{Iw}_{H^+}(\gamma g)^{(\operatorname{Re} s)\varpi_P} \operatorname{Iw}_{H^+}(\gamma g)^{nD\varpi_P} \\
= &C_1 \operatorname{Iw}_{H^+}(\gamma g)^{s_0\varpi_P}.
\end{aligned}$$

7. Ample parabolic subgroups

Proposition

Assume that G is infinite dimensional.

If $\langle \alpha_i, \alpha^\vee \rangle \leq 0$ *for any* $\alpha_i \in \Delta$, $\alpha \in \Phi'_w$ *where* $w^{-1}\alpha_i > 0$,

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then every maximal parabolic P with a finite dimensional Levi is ample.

- The condition in the above proposition implies that the group G satisfies Property $(\star)$.

- If G is finite dimensional, then G does not have any ample parabolic subgroup.

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- If G is a rank 2 hyperbolic group, then every maximal parabolic is ample.
- Feingold–Frenkel algebra: both maximal parabolic subgroups with finite dimensional Levi are ample.

Thank You