Some isoperimetric inequalities on $\mathbb{R}^N$ with respect to weights $|x|^\alpha$

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Geometric and Analytic Inequalities

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Classical Isoperimetric Problem

Enclose given volume with minimum perimeter

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Minimize $\mathbf{P}(\boldsymbol{\Omega})$ *among measurable sets* $\Omega \subset \mathbb{R}^N$ *such that*

$$\mathbf{0} < \mathbf{V}(\boldsymbol{\Omega}) = \mathbf{m} < +\infty$$

Isoperimetric Region and Isoperimetric inequality

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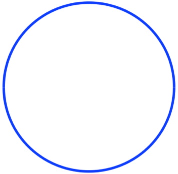

$\Omega^\star$ ball such that $\mathbf{V}(\boldsymbol{\Omega}) = \mathbf{m}$

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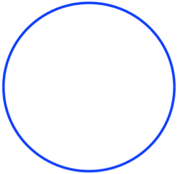

$\Omega^\star$ ball such that $\mathbf{V}(\mathbf{\Omega}) = \mathbf{m}$

$$\mathbf{P}(\mathbf{\Omega}) \geq \mathbf{P}(\mathbf{\Omega}^\star)$$

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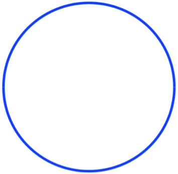

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$$\mathbf{P}(\boldsymbol{\Omega}) \geq \mathbf{P}(\boldsymbol{\Omega}^\star)$$

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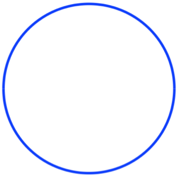

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$$\mathbf{P}(\boldsymbol{\Omega}) \geq \mathbf{P}(\boldsymbol{\Omega}^\star)$$

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Weighted Isoperimetric Problem

DENSITY Ψ is a positive continuous function

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$$\mathbf{dP_{\Psi} = \Psi dH^{n-1}}$$

$$\mathbf{P_{\Psi}(\Omega)} = \int_{\partial\Omega} \Psi(x)\, H_{N-1}(dx)$$

Weighted Perimeter

$$\mathbf{dV_{\Psi} = \Psi dx}$$

$$\mathbf{V_{\Psi}(\Omega)} = \int_{\Omega} \Psi(x)\, dx$$

Weighted Volume

Weighted Isoperimetric Problem

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Enclose given **weighted volume** with *weighted minimum perimeter*

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Enclose given **weighted volume** with *weighted minimum perimeter*

Minimize $\mathbf{P_{\Psi}(\Omega)}$ *among all smooth sets* $\Omega \subset \mathbb{R}^N$ *such that*
$0 < \mathbf{V_{\Psi}(\Omega) = m} < +\infty$

Look for $\Omega^\star$ such that

$$\mathbf{P_{\Psi}(\Omega)} \geq \mathbf{P_{\Psi}(\Omega^\star)}$$

Examples of *Weighted* Isoperimetric Problem

$$\mathbf{dP_\Psi = \Psi dH^{n-1}}, \qquad \mathbf{dV_\Psi = \Psi dx}$$

$$\mathbf{P_\Psi(\Omega)} \geq \mathbf{P_\Psi(\Omega^\star)}$$

$$\mathbf{\Omega^\star} = ??$$

GAUSSIAN DENSITY

$\Psi(x) = (2\pi)^{-N/2}\exp(-|x|^2/2)$

Sudakov, Tsirel'son, Borell, 1975

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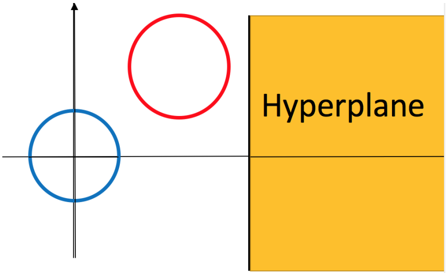

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$$\mathbf{dP_\Psi = \Psi dH^{n-1}}, \qquad \mathbf{dV_\Psi = \Psi dx}$$

$$\mathbf{P_\Psi(\Omega) \geq P_\Psi(\Omega^\star)}$$

$\mathbf{\Omega^\star}$ is an hyperplane

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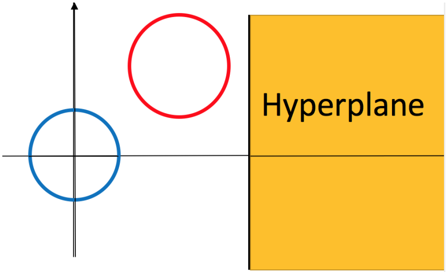

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$$\mathbf{\Omega^\star} = ??$$

LOG-CONVEX DENSITY

$\Psi(x) = \exp(f(|x|))$

f positive, smooth, convex

Chambers, 2013

Examples of *Weighted* Isoperimetric Problem

$$\mathbf{dP_\Psi = \Psi dH^{n-1}}, \qquad \mathbf{dV_\Psi = \Psi dx}$$

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$\mathbf{\Omega^\star}$ is a ball centered in the origin

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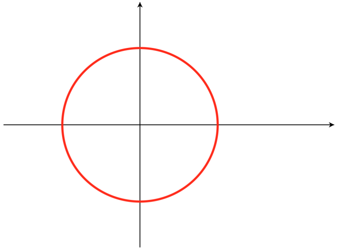

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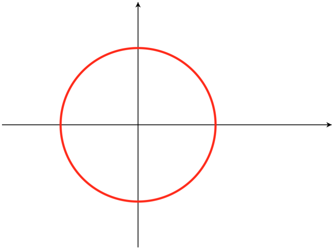

Borel, 1984
Bayle, Canete, Morgan, Rosales, 2008
Brock, M., Posteraro, 2013

Examples of *Weighted* Isoperimetric Problem

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$\Psi(x) = |x|^\alpha$,

$\alpha > 0$

Examples of *Weighted* Isoperimetric Problem

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$$\mathbf{P_{\Psi}(\Omega) \geq P_{\Psi}(\Omega^{\star})}$$

$\mathbf{\Omega^{\star}}$ is a ball through the origin

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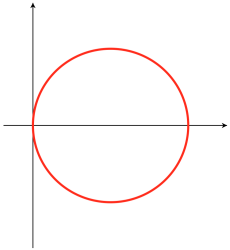

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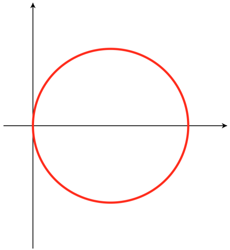

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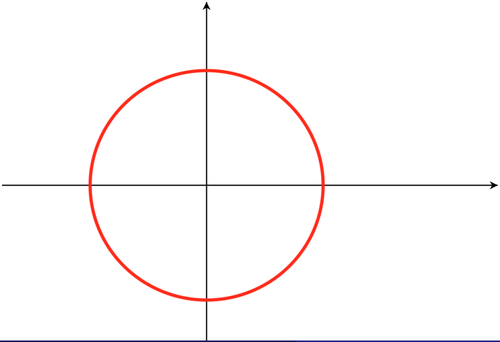
Weighted Isoperimetric Problem *with different densities*

Minimize $\mathbf{P_k}(\boldsymbol{\Omega})$ *among all smooth sets* $\Omega \subset \mathbb{R}^N$ *such that* $\mathbf{V}_\alpha(\boldsymbol{\Omega}) = \mathbf{m}$

$$\mathbf{P_k}(\boldsymbol{\Omega}) = \int_{\partial\Omega} |x|^k \, H_{N-1}(dx) \, , \qquad \mathbf{V}_\alpha(\boldsymbol{\Omega}) = \int_\Omega |\mathbf{x}|^\alpha \, dx$$



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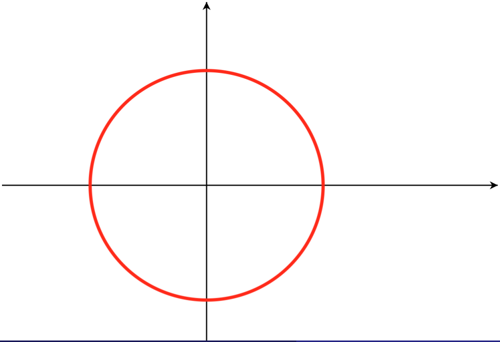

$\mathbf{\Omega^\star}$ is a ball centered in the origin

Find $\mathbf{k}\,,\alpha$
such that the minimum weighted perimeter
is attained for $\mathbf{\Omega^\star}$

How to formulate the *Weighted* Isoperimetric Inequality

$$dP_k = |x|^k dH^{n-1}\,, \qquad\qquad dV_\alpha = |x|^\alpha dx$$

Ω smooth set such that $V_\alpha(\Omega) = m$

$$P_k(\Omega) \geq P_k(\Omega^\star)$$

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$$P_k(\Omega) \geq P_k(\Omega^\star)$$

$$\mathcal{R}_{k,\alpha}(\Omega) := \frac{P_k(\Omega)}{(V_\alpha(\Omega))^{(k+N-1)/(\alpha+N)}} \geq \frac{P_k(\Omega^\star)}{(V_\alpha(\Omega^\star))^{(k+N-1)/(\alpha+N)}}$$

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$$\inf_\Omega \frac{P_k(\Omega)}{(V_\alpha(\Omega))^{(k+N-1)/(\alpha+N)}} \equiv C$$

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$$\inf_\Omega \frac{P_k(\Omega)}{(V_\alpha(\Omega))^{(k+N-1)/(\alpha+N)}} \equiv C = C^\star.$$

Our aim

- We look for values of k and α, such that

$$\int_{\partial\Omega} |x|^k \, H_{N-1}(dx) \geq C^\star \left(\int_\Omega |x|^\alpha \, dx \right)^{(k+N-1)/(\alpha+N)},$$

  for all smooth sets Ω in $\mathbb{R}^N$, where

$$C^\star := (N\omega_N)^{(\alpha-k+1)/(\alpha+N)} \cdot (\alpha+N)^{(k+N-1)/(\alpha+N)}.$$

- Equality holds for every ball centered at the origin.

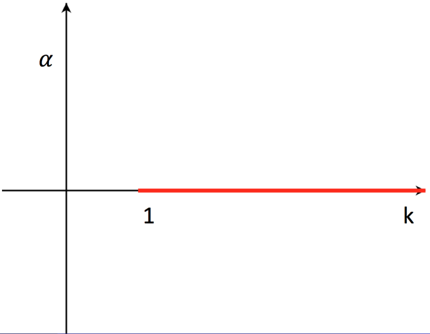
Some references

$\alpha = 0,\ k \geq 1,$ $\quad dP_k = |x|^k dH^{n-1}, \quad \mathbf{dV_\alpha = |x|^\alpha dx}$

$$P_k(\Omega) = \int_{\partial\Omega} |x|^k \, dH^{n-1}(dx) \geq P_k(\Omega^\star) = \int_{\partial\Omega^\star} |x|^k \, dH^{n-1}(dx)$$

for all smooth sets Ω in $\mathbb{R}^N$ such that $0 < V_0(\Omega) = V_0(\Omega^\star) < +\infty$

- Betta, Brock, M., Posteraro, 1999



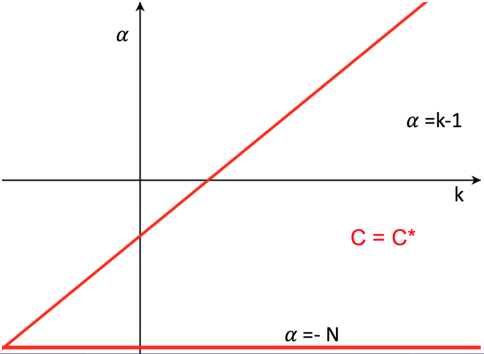
Some references

$N \geq 1, \alpha \leq k-1, \alpha > -N$ $\quad dP_k = |x|^k dH^{n-1}, \quad dV_\alpha = |x|^\alpha dx$

$$P_k(\Omega) \geq P_k(\Omega^\star)$$

for all smooth sets Ω in $\mathbb{R}^N$ such that $0 < V_\alpha(\Omega) < +\infty$

-Howe, 2015
-A. Diaz, N. Harman, S. Howe, D. Thompson, 2012 (case $N = 2$)



Some references

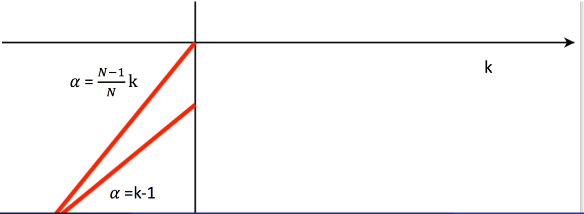
$N \geq 2$, $k-1 \leq \alpha$ and $\alpha \frac{N-1}{N} \leq k \leq 0$

$dP_k = |x|^k dH^{n-1}$, $\quad dV_\alpha = |x|^\alpha dx$

$$P_k(\Omega) \geq P_k(\Omega^\star)$$

for all smooth sets Ω in $\mathbb{R}^N$ such that $0 < V_\alpha(\Omega) < +\infty$

- N. Chiba, T. Horiuchi, 2015
- A. Diaz, N. Harman, S. Howe, D. Thompson, 2012 (case $N = 2$)



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- G. Csatò, 2015
$N = 2$, $\alpha = 0$, $0 \le k \le 1$

- V. Maz'ja, 2003
$\alpha < 0$, $k = 0$

A necessary condition for existence of minimizers

$$\inf_{\Omega} \frac{P_k(\Omega)}{(V_\alpha(\Omega))^{(k+N-1)/(\alpha+N)}} \equiv C > 0$$

$$\Downarrow$$

$$\alpha \leq \frac{N}{N-1} k$$

Alvino, Brock, Chiacchio, M., Posteraro, 2016

A necessary condition for existence of minimizers

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Alvino, Brock, Chiacchio, M., Posteraro, 2016

Assume $te_1 = (t, 0, \ldots, 0)$ and $\alpha > \frac{N}{N-1} k$

$$\mathcal{R}_{k,\alpha}(B_1(te_1)) = \frac{P_k(B_1(te_1))}{(V_\alpha(B_1(te_1)))^{(k+N-1)/(\alpha+N)}} \leq D \frac{t^k}{t^{\alpha(k+N-1)/(\alpha+N)}},$$

$$\Downarrow$$

$$\lim_{t\to\infty} \mathcal{R}_{k,\alpha}(B_1(te_1)) = 0.$$

A necessary condition for radially of minimizers

$$C = C^{\star}$$

$$\Downarrow$$

$$\boxed{\alpha \leq k - 1 + \frac{N-1}{k+N-1}}$$

Alvino, Brock, Chiacchio, M., Posteraro, 2016

Weighted Isoperimetric Inequality: Main results

- $N \geq 3,$

$$0 \leq k-1 \leq \alpha, \qquad \frac{1}{\alpha+N} \geq \frac{1}{k+N-1} - \frac{(N-1)^2}{N(k+N-1)^3}$$

$$\int_{\partial\Omega} |x|^k \, H_{N-1}(dx) \geq C^\star \left(\int_\Omega |x|^\alpha \, dx \right)^{(k+N-1)/(\alpha+N)},$$

for all smooth sets Ω in $\mathbb{R}^N$, where

$$C^\star := (N\omega_N)^{(\alpha-k+1)/(\alpha+N)} \cdot (\alpha+N)^{(k+N-1)/(\alpha+N)}.$$

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- Equality holds for every ball centered at the origin.

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- Equality holds for every ball centered at the origin.
- For certain values of k and α, if equality holds, Ω is a ball centered at the origin.

Alvino, Brock, Chiacchio, M., Posteraro, 2016

Weighted Isoperimetric Inequality: Main results

- $N = 2,$ $0 \le k - 1 \le \alpha,$ and

  either $\alpha \le 0 \le k \le \frac{1}{3}$ or $\frac{1}{3} \le k$ and $\frac{1}{\alpha+2} \ge \frac{1}{k+1} - \frac{16}{27(k+1)^3}$

$$\int_{\partial\Omega} |x|^k \, H_{N-1}(dx) \ge C^\star \left(\int_\Omega |x|^\alpha \, dx \right)^{(k+1)/(\alpha+2)},$$

  for all smooth sets Ω in $\mathbb{R}^2$, where

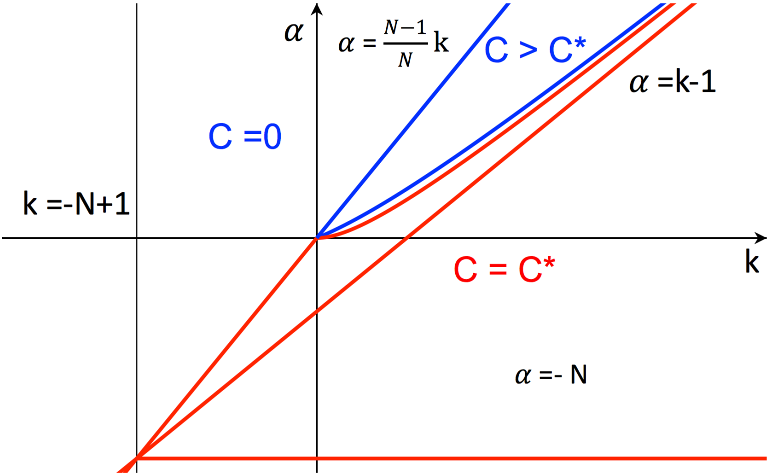
$$C^\star := (2\pi)^{(\alpha-k+1)/(\alpha+2)} \cdot (\alpha+2)^{(k+1)/(\alpha+2)}.$$

- Equality holds for every ball centered at the origin.
- For certain values of k and α, if equality holds, Ω is a ball centered at the origin.

Alvino, Brock, Chiacchio, M., Posteraro, 2016

Weighted Isoperimetric Inequality: values of k and α

$dP_k = |x|^k dH^{n-1}$, $dV_\alpha = |x|^\alpha dx$ $N \geq 3,$



Classical Polya-Szegö Principle

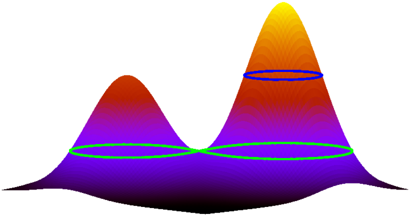
$$\int_{\mathbb{R}^N} |\nabla u|^p \, dx \geq \int_{\mathbb{R}^N} |\nabla u^\star|^p \, dx, \qquad u \in W_0^{1,p}(\mathbb{R}^N)$$

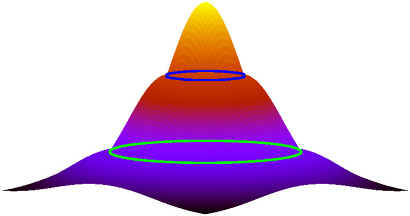
where
$p \in [1, +\infty)$,
$u^\star$ be the Schwarz symmetrization of u.

Schwarz symmetrization of u

u

$u^{\#}$ **spherically decreasing rearrangement**





Applications: Polya-Szegö Principle

$$dP_k = |x|^k dH^{n-1}\,, dV_\alpha = |x|^\alpha dx$$

Weighted isoperimetric inequality with $k = a \quad \alpha = 0$

$$\Downarrow$$

$$\int_{\mathbb{R}^N} |\nabla u|^p \, |x|^{ap} \, dx \geq \int_{\mathbb{R}^N} |\nabla u^\star|^p \, |x|^{ap} \, dx, \qquad u \in C_0^\infty(\mathbb{R}^N)$$

where
$p \in [1, +\infty)$, $a \geq 0$,
$u^\star$ be the Schwarz symmetrization of u.

Alvino, Brock, Chiacchio, M. Posteraro, 2016
Talenti, 1993,.........

Applications: Best Constants in some Caffarelli-Kohn-Nirenberg inequalities

- CKN - inequalities

$$\left(\int_{\mathbb{R}^N} |v|^q \, |x|^{bq} \, dx\right)^{\frac{p}{q}} S \le \int_{\mathbb{R}^N} |\nabla v|^p \, |x|^{ap} \, dx \,, \quad v \in C_0^\infty(\mathbb{R}^N) \setminus \{0\}$$

Let p, q, a, b be real numbers such that

$$1 \le p \le q \begin{cases} \le \frac{Np}{N-p} & \text{if } p < N \\ < +\infty & \text{if } p \ge N \end{cases},$$

$$a > 1 - \frac{N}{p}, \quad \text{and}$$

$$b = N\left(\frac{1}{p} - \frac{1}{q}\right) + a - 1.$$

-Caffarelli, Kohn, Nirenberg, 1984, - Catrina, Wang, 2001

Best Constants in some Caffarelli-Kohn-Nirenberg inequalities

$$E(v) := \frac{\displaystyle\int_{\mathbb{R}^N} |x|^{ap} |\nabla v|^p \, dx}{\left(\displaystyle\int_{\mathbb{R}^N} |x|^{bq} |v|^q \, dx\right)^{p/q}}, \quad v \in C_0^\infty(\mathbb{R}^N) \setminus \{0\},$$

Best Constants
in some Caffarelli-Kohn-Nirenberg inequalities

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$$S := \inf\{E(v) : \, v \in C_0^\infty(\mathbb{R}^N) \setminus \{0\}\},$$

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$$S^\star := \inf\{E(v) : \; v \in C_0^\infty(\mathbb{R}^N) \setminus \{0\}, \; v \; \text{radial} \; \}.$$

Best Constants in some Caffarelli-Kohn-Nirenberg inequalities

$$E(v) := \frac{\displaystyle\int_{\mathbb{R}^N} |x|^{ap} |\nabla v|^p \, dx}{\left(\displaystyle\int_{\mathbb{R}^N} |x|^{bq} |v|^q \, dx \right)^{p/q}}, \quad v \in C_0^\infty(\mathbb{R}^N) \setminus \{0\},$$

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$$S^\star := \inf\{E(v) : v \in C_0^\infty(\mathbb{R}^N) \setminus \{0\}, \ v \ \text{radial} \ \}.$$

$$\boxed{S \le S^\star}$$

- Hardy - Sobolev inequality, that is $p = q$,

$$S \int_{\mathbb{R}^N} |v|^p \, |x|^{bp} \, dx \le \int_{\mathbb{R}^N} |\nabla v|^p \, |x|^{ap} \, dx.$$

and

$$\boxed{S = S^\star = \left(\frac{N}{p} - 1 + a\right)^p .}$$

Hardy, 1919

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  and

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  Hardy, 1919

- $1 < p < N$, $q = p*$, $a \le 0$,

$$S = S^\star$$

  Horiouchi-Kumlin, 2012

Problem

Assume

$$N \geq 2 \;\text{ and }\; 1 < p < q < p^*$$

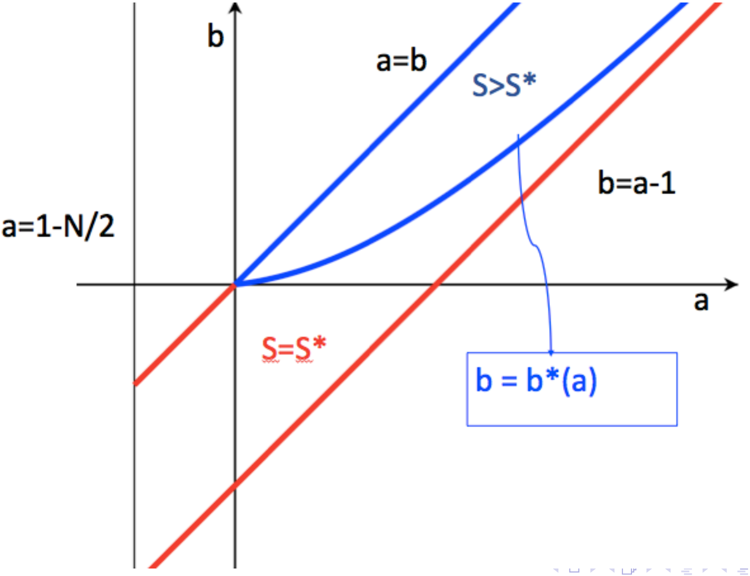
Find (a, b) such that

$$S = S^\star$$

or

$$S < S^\star$$

Dalbeaux, Esteban, Loss, 2015 ($p = 2$, $N \geq 3$)



$a_0 = a_0(p,q,N)$ as the unique number in $\left(1-\frac{N}{p}, +\infty\right)$ such that

$$\boxed{\left(\frac{N}{p}-1+a_0\right)^2 = \frac{(N-1)^2}{N\left(\frac{1}{p}-\frac{1}{q}\right)\cdot\left(1-\frac{q}{p}+q\right)^2}}$$

$$\boxed{b = N\left(\frac{1}{p}-\frac{1}{q}\right)+a-1}$$

$$\boxed{a \leq a_0}$$

$$\Downarrow$$

$$S = S^\star$$

- Alvino, Brock, Chiacchio,Posteraro, 2016
- Felli, Schneider, 2003
- Caldiroli, Musina, 2013.

Idea of the proof

Polya-Sžego principle

$$\Downarrow$$

$$E(v) := \frac{\displaystyle\int_{\mathbb{R}^N} |x|^{ap} |\nabla v|^p \, dx}{\left(\displaystyle\int_{\mathbb{R}^N} |x|^{bq} |v|^q \, dx\right)^{p/q}} \geq E(v^\star) := \frac{\displaystyle\int_{\mathbb{R}^N} |x|^{ap} |\nabla v^\star|^p \, dx}{\left(\displaystyle\int_{\mathbb{R}^N} |x|^{bq} |v^\star|^q \, dx\right)^{p/q}}$$

$$\Downarrow$$

$$S \geq S^\star$$

$$\Downarrow$$

$$S = S^\star$$

An eigenvalue problem: sharp lower bound

The eigenvalue problem

$$\begin{cases} -\mathrm{div}(|\nabla u|^{p-2}\nabla u) = \lambda |x|^{-\beta p}|u|^{p-2}u & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

where Ω is a bounded domain in $\mathbb{R}^N$, $1 < p < N$ and $0 \leq \beta < 1$.

$$\lambda_1(\Omega) = \min\left\{ \frac{\displaystyle\int_\Omega |\nabla\varphi|^p dx}{\displaystyle\int_\Omega |x|^{-\beta p}|\varphi(x)|^p dx} : \ \varphi \in W_0^{1,p}(\Omega) \setminus \{0\} \right\}.$$

Sharp lower bound for $\lambda_1(\Omega)$

$$\lambda_1(\Omega) \geq \lambda_1(\Omega^\star)$$

$\Omega^\star$ is the ball with center in the origin s.t. $V_{-\beta p}(\Omega) = V_{-\beta p}(\Omega^\star)$

- there exists a sequence $\{u_n\} \subset C_0^\infty(\mathbb{R}^N) \setminus \{0\}$ such that

$$\lim_{n\to\infty} \int_{\mathbb{R}^N} |x|^k |\nabla u_n| \, dx = \int_{\partial\Omega} |x|^k \, H_{N-1}(dx) \equiv P_k(\Omega),$$

$$\lim_{n\to\infty} \int_{\mathbb{R}^N} |x|^\alpha |u_n|^{(\alpha+N)/(k+N-1)} \, dx = \int_\Omega |x|^\alpha \, dx = V_\alpha(\Omega).$$

$$\Downarrow$$

$$C \equiv \inf_{\Omega} \frac{P_k(\Omega)}{(V_\alpha(\Omega))^{(k+N-1)/(\alpha+N)}} =$$

$$= \inf_{\substack{u\in C_0^1\\ \neq 0}} \frac{\displaystyle\int_{\mathbb{R}^N} |x|^k |\nabla u| \, dx}{\left(\displaystyle\int_{\mathbb{R}^N} |x|^l |u|^{(l+N)/(k+N-1)} \, dx\right)^{(k+N-1)/(l+N)}}.$$

Weighted Isoperimetric Inequalities: main tools

- there exists a sequence $\{u_n\} \subset C_0^\infty(\mathbb{R}^N) \setminus \{0\}$ such that

$$\lim_{n\to\infty} \int_{\mathbb{R}^N} |x|^k |\nabla u_n|\, dx = \int_{\partial\Omega} |x|^k \, H_{N-1}(dx) \equiv P_k(\Omega),$$

$$\lim_{n\to\infty} \int_{\mathbb{R}^N} |x|^\alpha |u_n|^{(\alpha+N)/(k+N-1)}\, dx = \int_{\Omega} |x|^\alpha \, dx = V_\alpha(\Omega).$$

$$\Downarrow$$

$$C \equiv \inf_{\Omega} \frac{P_k(\Omega)}{(V_\alpha(\Omega))^{(k+N-1)/(\alpha+N)}} =$$

$$= \inf_{u\in C_0^1} \frac{\displaystyle\int_{\mathbb{R}^N} |x|^k |\nabla u|\, dx}{\left(\displaystyle\int_{\mathbb{R}^N} |x|^\alpha |u|^{(\alpha+N)/(k+N-1)}\, dx\right)^{(k+N-1)/(\alpha+N)}}.$$

Weighted Isoperimetric Inequalities: main tools

$$\mathcal{Q}_{k,\alpha}(u) = \frac{\displaystyle\int_{\mathbb{R}^N} |x|^k |\nabla u| \, dx}{\left(\displaystyle\int_{\mathbb{R}^N} |x|^l |u|^{(l+N)/(k+N-1)} \, dx \right)^{(k+N-1)/(l+N)}}.$$

- Change of variables $z := |y| = r^{\frac{k+N-1}{N-1}}$,
- Interpolation argument

$$\int_{\mathbb{R}^N} |x|^k |\nabla_x u| \, dx = \int_{\mathscr{S}^{N-1}} \int_0^{+\infty} z^{N-1} \sqrt{v_z^2 + \frac{|\nabla_\theta v|^2}{z^2} \frac{(N-1)^2}{(k+N-1)^2}} \, dz \, d\theta.$$

and

$$t \longmapsto \log \left(\int_{\mathscr{S}^{N-1}} \int_0^{+\infty} z^{N-1} \sqrt{v_z^2 + t \frac{|\nabla_\theta v|^2}{z^2}} \, dz \, d\theta \right)$$

is concave.

- For every $A \in \left[0, \frac{(N-1)^2}{(k+N-1)^2}\right]$,

$$\mathcal{Q}_{k,l,N}(u) \geq \left(\frac{k+N-1}{N-1}\right)^{\frac{k+N-1}{l+N}} \cdot \frac{\left(\int_{\mathbb{R}^N} |\nabla_y v| \, dy\right)^A \cdot \left(\int_{\mathbb{R}^N} |v_z| \, dy\right)^{1-A}}{\left(\int_{\mathbb{R}^N} |y|^{\frac{l(N-1)-kN}{k+N-1}} v^{\frac{l+N}{k+N-1}} \, dy\right)^{\frac{k+N-1}{l+N}}}.$$

- Starshaped symmetrization