Symmetry breaking for a problem in optimal insulation

Giuseppe Buttazzo
Dipartimento di Matematica
Università di Pisa
buttazzo@dm.unipi.it
http://cvgmt.sns.it

"Geometric and Analytic Inequalities"
Banff, July 10–15, 2016

how termically efficient is this building?

is this well designed?

Joint work

D. Bucur, G. Buttazzo, C. Nitsch
Symmetry breaking for a problem in optimal insulation.
To appear on J. Mat. Pures Appl.

available at:
`http://cvgmt.sns.it`
`http://arxiv.org`

Let Ω be a given domain of $\mathbf{R}^d$ that we want to thermically insulate adding around its boundary a given amount of insulating material as a thin layer (variable thickness).

• How to measure the efficiency of the design?

• Is there an optimal way to arrange the insulating material around $\partial\Omega$?

• If also Ω may vary, is there an optimal domain Ω among the ones of prescribed Lebesge measure?

Two different criteria are possible.

1. Put in Ω a heat source, for instance $f = 1$, wait enough time, and then measure the (average) temperature.

2. Fix an initial temperature u_0, no heat source, and see how quick the temperature decays in time.

We put now the problems in a precise mathematical form.

• Assume that in Ω the conductivity coefficient is 1, while it is δ in the insulating material.

• Describe the shape of the insulator as

$$\Sigma_\varepsilon = \Big\{\sigma + t\nu(\sigma) \;:\; \sigma \in \partial\Omega,\ 0 \le t < \varepsilon h(\sigma)\Big\}.$$

where the function h takes into account the variable thickness. The temperature $u(t, x)$ is assumed to vanish outside $\Omega \cup \Sigma_\varepsilon$.

• Denote by $f \in L^2(\Omega)$ the heat sources.

Then as $t \to \infty$ the temperature $u(t,x)$ solves the stationary elliptic problem

$$\begin{cases} -\Delta u = f & \text{in } \Omega \\ -\Delta u = 0 & \text{in } \Sigma_\varepsilon \\ u = 0 & \text{on } \partial(\Omega \cup \Sigma_\varepsilon) \\ \dfrac{\partial u^-}{\partial \nu} = \delta \dfrac{\partial u^+}{\partial \nu} & \text{on } \partial\Omega. \end{cases}$$

or equivalently minimizes on $H_0^1(\Omega \cup \Sigma_\varepsilon)$ the functional

$$F_{\varepsilon,\delta}(u) = \frac{1}{2}\int_\Omega |\nabla u|^2\,dx + \frac{\delta}{2}\int_{\Sigma_\varepsilon} |\nabla u|^2\,dx - \int_\Omega f u\,dx.$$

The asymptotic behavior of the functionals $F_{\varepsilon,\delta}$ as ε and δ tend to zero was studied in a great generality (general functionals) in

E. Acerbi, G. Buttazzo: *Reinforcement problems in the calculus of variations.* Ann. Inst. H. Poincaré Anal. Non Linéaire, **3** (1986), 273–284.

previous analysis in the Dirichlet energy case:

H. Brezis, L. Caffarelli, A. Friedman: *Reinforcement problems for elliptic equations and variational inequalities.* Ann. Mat. Pura Appl., **123** (1980), 219–246.

The only interesting case is $\varepsilon \approx \delta$ in which the Γ-limit functional is (we stress the dependence on h)

$$F(u,h) = \frac{1}{2}\int_\Omega |\nabla u|^2\,dx + \frac{1}{2}\int_{\partial\Omega} \frac{u^2}{h}\,d\mathcal{H}^{d-1} - \int_\Omega f u\,dx.$$

Therefore the stationary temperature u solves the minimum problem

$$E(h) = \min\left\{F(u,h)\ :\ u \in H^1(\Omega)\right\}$$

or equivalently the PDE (of Robin type)

$$\begin{cases} -\Delta u = f & \text{in } \Omega \\ h\dfrac{\partial u}{\partial \nu} + u = 0 & \text{on } \partial\Omega. \end{cases}$$

We want to find the function h which provides the best insulating performances, once the total amount of insulator is fixed, that is we consider the class

$$\mathcal{H}_m = \left\{ h : \partial\Omega \to \mathbf{R},\ h \geq 0,\ \int_{\partial\Omega} h \, d\mathcal{H}^{d-1} = m \right\}.$$

In this first category of problems we minimize the total energy $E(h)$, which can be written in terms of the solution u of the PDE above, multiplying both sides of the PDE by u and integrating by parts:

$$E(h) = -\frac{1}{2} \int_\Omega f u \, dx.$$

Thus, if the heat sources are uniformly distributed (i.e. $f = 1$), minimizing $E(h)$ corresponds to maximizing the average temperature in Ω. Therefore our first optimization problem can be written as

$$\min\Big\{E(h)\ :\ h\in\mathcal{H}_m\Big\}.$$

This problem was studied in

G. Buttazzo: *Thin insulating layers: the optimization point of view.* Oxford University Press, Oxford (1988), 11–19.

The second optimization problem aims to minimize the decay in time of the temperature, once an initial condition is fixed, with no heat sources.

It is well known that, by a Fourier analysis, the long time behavior of the temperature $u(t,x)$ goes as $e^{-t\lambda(h)}$, where $\lambda(h)$ is the first eigenvalue of the operator $\mathcal{A}$ written in a weak form as

$$\langle \mathcal{A}u, \phi \rangle = \int_\Omega \nabla u \nabla \phi \, dx + \int_{\partial\Omega} \frac{u\phi}{h} \, d\mathcal{H}^{d-1}.$$

The eigenvalue $\lambda(h)$ is given by the Rayleigh quotient

$$\lambda(h) = \inf_{u \in H^1(\Omega)} \left\{ \frac{\int_\Omega |\nabla u|^2\,dx + \int_{\partial\Omega} u^2/h\,d\mathcal{H}^{d-1}}{\int_\Omega u^2\,dx} \right\}$$

and therefore, the second optimization problem we deal with is

$$\min \left\{ \lambda(h) \ : \ h \in \mathcal{H}_m \right\}.$$

We will see that the two problems behave in a quite different way.

This second category of problems was proposed in

A. Friedman: *Reinforcement of the principal eigenvalue of an elliptic operator.* Arch. Rational Mech. Anal., **73** (1980), 1–17.

A partial answer (valid only in some regimes of large values of m) is in

S.J. Cox, B. Kawohl, P.X. Uhlig: *On the optimal insulation of conductors.* J. Optim. Theory Appl., **100** (1999), 253–263.

Problem 1: energy optimization

The problem we deal with is

$$\min_{h\in\mathcal{H}_m} \min_{u\in H^1(\Omega)} \left\{ \frac{1}{2}\int_\Omega |\nabla u|^2\,dx + \frac{1}{2}\int_{\partial\Omega} \frac{u^2}{h}\,d\mathcal{H}^{N-1} - \int_\Omega f u\,dx \right\}$$

which, interchanging the two min, gives

$$\min_{u\in H^1(\Omega)} \min_{h\in\mathcal{H}_m} \left\{ \frac{1}{2}\int_\Omega |\nabla u|^2\,dx + \frac{1}{2}\int_{\partial\Omega} \frac{u^2}{h}\,d\mathcal{H}^{N-1} - \int_\Omega f u\,dx \right\}.$$

The minimum with respect to h is easy to compute explicitly and, for a fixed $u \in H^1(\Omega)$, is reached for

$$h = m\frac{|u|}{\int_{\partial\Omega} |u|\,d\mathcal{H}^{d-1}}\,.$$

Then, problem 1 can be rewritten as

$$\min_{u\in H^1(\Omega)}\left\{\frac{1}{2}\int_\Omega |\nabla u|^2\,dx+\frac{1}{2m}\Big(\int_{\partial\Omega}|u|\,d\mathcal{H}^{d-1}\Big)^2-\int_\Omega f\,u\,dx\right\}.$$

The existence of a solution for this problem follows by the Poincaré-type inequality

$$\int_\Omega u^2\,dx\le C\left[\int_\Omega |\nabla u|^2\,dx+\Big(\int_{\partial\Omega}|u|\,d\mathcal{H}^{d-1}\Big)^2\right]$$

which implies the coercivity of the functional above. The solution is also unique, thanks to the result below.

Theorem Assume Ω is connected. Then the functional

$$F(u) = \frac{1}{2}\int_\Omega |\nabla u|^2\,dx + \frac{1}{2m}\Big(\int_{\partial\Omega}|u|\,d\mathcal{H}^{d-1}\Big)^2$$

is strictly convex on $H^1(\Omega)$, hence for every $f \in L^2(\Omega)$ the minimization problem above admits a unique solution.

Corollary By uniqueness, if $\Omega = B_R$ in $\mathbf{R}^d$ and $f = 1$ the optimal solution u is radial:

$$u(r) = \frac{R^2 - r^2}{2d} + c \qquad \Big(c = \frac{m}{d^2\omega_d R^{d-2}}\Big).$$

The optimal thickness h_{opt} is then constant.

If Ω is not connected the optimal insulation strategy is different. Let $\Omega = B_{R_1} \cup B_{R_2}$ in $\mathbf{R}^d$ (union of two disjoint balls), and $f = 1$.

• If $R_1 = R_2 = R$ any choice of h constant around B_{R_1} and on B_{R_2} is optimal;

• if $R_1 \neq R_2$ then the optimal choice is to concentrate all the insulator around the largest ball, with constant thickness, leaving the smallest ball unprotected.

Problem 2: eigenvalue optimization

The problem we deal with is now

$$\min_{h\in\mathcal{H}_m}\ \min_{u\in H^1(\Omega)}\left\{\frac{\int_\Omega |\nabla u|^2\,dx+\int_{\partial\Omega} u^2/h\,d\mathcal{H}^{d-1}}{\int_\Omega u^2\,dx}\right\}$$

and again, interchanging the two min:

$$\min_{u\in H^1(\Omega)}\ \min_{h\in\mathcal{H}_m}\left\{\frac{\int_\Omega |\nabla u|^2\,dx+\int_{\partial\Omega} u^2/h\,d\mathcal{H}^{d-1}}{\int_\Omega u^2\,dx}\right\}.$$

The min in h is reached again for

$$h=m\frac{|u|}{\int_{\partial\Omega}|u|\,d\mathcal{H}^{d-1}}$$

and so we obtain the optimization problem

$$\min_{u \in H^1(\Omega)} \left\{ \frac{\int_\Omega |\nabla u|^2\,dx + \frac{1}{m}\Big(\int_{\partial\Omega} |u|\,d\mathcal{H}^{d-1}\Big)^2}{\int_\Omega u^2\,dx} \right\}.$$

Again, the existence of a solution $\bar{u}$ easily follows from the direct methods of the calculus of variations, and the optimal h_{opt} is

$$h_{opt} = m\frac{\bar{u}}{\int_{\partial\Omega} \bar{u}\,d\mathcal{H}^{d-1}}.$$

Question: If Ω is a ball, is h_{opt} constant?

Note that the convexity of the auxiliary problem does not occur anymore.

Theorem Let $\Omega = B_R$. There exists $m_0 > 0$ such that:
• if $m > m_0$ $\bar{u}$ is radial, hence h_{opt} is constant;
• if $m < m_0$ $\bar{u}$ is not radial, hence h_{opt} is not constant.

Let us give a quick idea of the proof of the symmetry breaking (the first part is more involved).

Set for every $m > 0$

$$J_m(u) = \frac{\int_\Omega |\nabla u|^2\,dx + \frac{1}{m}\Big(\int_{\partial\Omega} |u|\,d\sigma\Big)^2}{\int_\Omega u^2\,dx}\ ;$$

$$\lambda_m = \min\Big\{J_m(u) \ : \ u \in H^1(\Omega)\Big\}\ ;$$

$$\lambda_N = \min\Big\{J_\infty(u) \ : \ u \in H^1(\Omega),\ \int_\Omega u\,dx = 0\Big\}\ ;$$

first nonzero Neumann eigenvalue of $-\Delta$;

$$\lambda_D = \min\Big\{J_\infty(u) \ : \ u \in H^1_0(\Omega)\Big\}\ ;$$

first Dirichlet eigenvalue of $-\Delta$.

Observe that λ_m is decreasing in m and

$$\lambda_m \to 0 \qquad \text{as } m \to \infty,$$
$$\lambda_m \to \lambda_D \qquad \text{as } m \to 0.$$

The Neumann eigenvalue λ_N is then in between and $\lambda_{m_0} = \lambda_N$ for a suitable m_0. This m_0 is the threshold value in the statement.

If $m < m_0$ assume by contradiction that $\bar{u}$ is radial; we take $\bar{u} + \varepsilon v$ as a test function, with v the first Neumann eigenfunction. We may take $\int_B \bar{u}^2\, dx = \int_B v^2\, dx = 1$.

We have that $\bar{u}$ and v are orthogonal, and also that $\int_{\partial B} v \, d\sigma = 0$. Then

$$\lambda_m = J_m(u) \leq J_m(u + \varepsilon v) = \frac{\lambda_m + \varepsilon^2 \lambda_N}{1 + \varepsilon^2}$$

which implies $\lambda_m \leq \lambda_N$ in contradiction to the fact that $\lambda_m > \lambda_N$ for $m < m_0$.

The conclusion for a circular domain is that the insulation giving the slowest decay of the temperature is by a constant thickness if we have enough insulating material. On the contrary, for a small amount of insulator, the best thickness in nonconstant.

• When the dimension $d = 1$ no symmetry breaking occurs. In fact when $d = 1$ the first nontrivial Neumann eigenvalue λ_N coincides with the first Dirichlet eigenvalue λ_D.

• The shape optimization problem related to problems 1 and 2:

$$\min\Big\{E(h,\Omega)\ :\ h\in\mathcal{H}_m,\ |\Omega|=M\Big\}$$
$$\min\Big\{\lambda(h,\Omega)\ :\ h\in\mathcal{H}_m,\ |\Omega|=M\Big\}$$

look very difficult and we do not have at the moment an existence result of an optimal shape.

It would be very interesting to prove (or disprove) that for both problems an optimal shape exists and that it is a ball in the first case while it is not a ball (for small m) in the second case.