On the growth of grid classes and staircases of permutations

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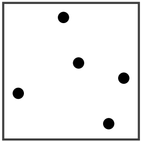
The Containment Order



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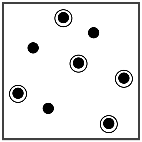

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- A downset in this order is a *permutation class*.
- Every permutation class $\mathcal{C}$ can be defined by the minimal set of permutations B it *avoids* (its *basis*):

$$\mathcal{C} = \mathrm{Av}(B) = \{\pi \,:\, \beta \nleqslant \pi \text{ for all } \beta \in B\}.$$

- $\mathcal{C}_n$ is the set of permutations in $\mathcal{C}$ of length n.
- The generating function of $\mathcal{C}$ is

$$\sum_{n \in \mathbb{N}} |\mathcal{C}_n| x^n = \sum_{\pi \in \mathcal{C}} x^{|\pi|}.$$

Growth Rates

- **If the limit exists**, the *growth rate* of $\mathcal{C}$ is

$$\mathrm{gr}(\mathcal{C}) = \lim_{n\to\infty} \sqrt[n]{|\mathcal{C}_n|}.$$

- Otherwise we settle for the *upper growth rate*,

$$\overline{\mathrm{gr}}(\mathcal{C}) = \limsup_{n\to\infty} \sqrt[n]{|\mathcal{C}_n|}.$$

The Marcus–Tardos Theorem. *Every proper permutation class has a finite upper growth rate.*

Sums and Skew Sums

$$\pi \oplus \sigma = \begin{array}{cc} & \boxed{\sigma} \\ \boxed{\pi} & \end{array} \qquad \pi \ominus \sigma = \begin{array}{cc} \boxed{\pi} & \\ & \boxed{\sigma} \end{array}$$

- The class $\mathcal{C}$ is *sum closed* if $\pi \oplus \sigma \in \mathcal{C}$ for all $\pi, \sigma \in \mathcal{C}$.
- If the class $\mathcal{C}$ is sum closed then

  $$|\mathcal{C}_m||\mathcal{C}_n| \leqslant |\mathcal{C}_{m+n}|,$$

  so $\{|\mathcal{C}_n|\}$ is supermultiplicative.
- By Fekete's Lemma, sum closed classes always have growth rates (Arratia 1999).
- Analogously, skew closed classes have growth rates.
- Thus all classes with singleton bases have growth rates.

Av(21)

- Enumeration: $1, 1, 1, 1, 1, \ldots$.
- Generating function: $1 + x + x^2 + \cdots = \dfrac{1}{1-x}$.
- Growth rate: 1.

$\mathrm{Av}(231, 312, 321)$

- Enumeration: $1, 2, 3, 5, 8, \ldots$ (empty permutation ignored — Fibonacci numbers).
- Generating function: $\dfrac{1}{1-x-x^2}$.
- Growth rate: $\phi \approx 1.62$ (golden ratio).

Av(321)

- Enumeration: 1, 2, 5, 14, 42, ... (Catalan numbers).
- Generating function: $\dfrac{1-\sqrt{1-4x}}{2x}$.
- Growth rate: 4.

Av(4231)

- Enumeration: 1, 2, 6, 23, 103, ... (only 36 terms known).
- Generating function: ?
- Growth rate: between 9.81 and 13.74, maybe around 11.60? (Bevan, Bóna, and Guttmann, improvements planned for 2017.)

$\mathrm{Av}(k \cdots 21)$

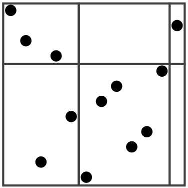
- Generating function: D-finite, due to Gessel 1990.
- Growth rate: $(k-1)^2$, due to Regev 1981.

$\mathrm{Av}(k \cdots 21)$

- Generating function: D-finite, due to Gessel 1990.
- Growth rate: $(k-1)^2$, due to Regev 1981.
  - Upper bound? Easy: the entries of $\pi \in \mathrm{Av}(k \cdots 21)$ can be partitioned in $k-1$ increasing subsequences. Form two words, one reading left-to-right, the other bottom-to-top.
  - **Lower bound?** One not-easy way was done by Bóna 2005.

The Defintion

The *grid class* of a matrix of permutation classes consists of all permutations which can be gridded so that the subpermutations in the cells lie in the respective classes.



$$\in \mathrm{Grid}\begin{pmatrix} \mathrm{Av}(12) & \emptyset & \mathrm{Av}(21) \\ \mathrm{Av}(21) & \mathrm{Av}(321) & \emptyset \end{pmatrix}$$

Question. *How does the growth rate of a grid class depend on those of its cells? (Assuming they have growth rates.)*

Answered by Bevan in 2015 for monotone cells.

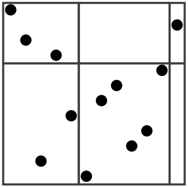
The Theorem

Theorem (Albert and V). *Let $\mathcal{M}$ be a $t \times u$ matrix of permutation classes, each with a proper growth rate, and define the matrix Γ of the same size by*

$$\Gamma_{k,\ell} = \sqrt{\mathrm{gr}(\mathcal{M}_{k,\ell})}.$$

The growth rate of $\mathrm{Grid}(\mathcal{M})$ *is equal to the greatest eigenvalue of $\Gamma^T\Gamma$ (or equivalently, of $\Gamma\Gamma^T$).*

A Brief Sketch



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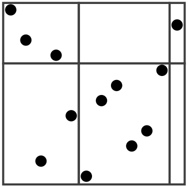

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A given permutation in $\mathrm{Grid}(\mathcal{M})$ has only polynomially many *griddings*, so we can count *gridded permutations* instead.

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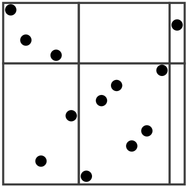

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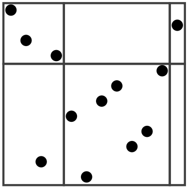

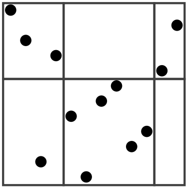

A Brief Sketch

The matrix A (of the same size as $\mathcal{M}$) is *admissible* if $A_{k,\ell} = 0$ whenever $\mathcal{M}_{k,\ell} = \emptyset$.

For admissible A, define

$$\mathrm{Grid}^{\sharp}_{A}(\mathcal{M}) = \begin{array}{c}\text{\# of gridded permutations in } \mathrm{Grid}(\mathcal{M}) \\ \text{with } A_{k,\ell} \text{ entries in cell } (k,\ell).\end{array}$$

For a given value of n there are only polynomial many admissible matrices which sum to n, so it suffices to find the admissible matrix which maximizes $|\mathrm{Grid}^{\sharp}_{A}(\mathcal{M})|$.

A Brief Sketch

$$|\mathrm{Grid}^{\sharp}_{A}(\mathcal{M})| = \begin{array}{l} \prod_{k=1}^{t} \binom{\sum A_{k,\bullet}}{A_{k,1},\dots,A_{k,u}} \\ \quad \times \prod_{\ell=1}^{u} \binom{\sum A_{\bullet,\ell}}{A_{1,\ell},\dots,A_{t,\ell}} \\ \quad \times \prod_{\substack{k,\ell \\ \mathcal{M}_{k,\ell \neq \emptyset}}} (\mathcal{M}_{k,\ell})_{A_{k,\ell}}. \end{array}$$

A Brief Sketch

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Fast-Forwarding to the End

- Using a compactness argument, we translate to maximizing a continuous function.
- We then apply Lagrange multipliers (actually to the logarithm of this function).
- This shows that the growth rate of $\mathrm{Grid}(\mathcal{M})$ is equal to the square of the largest singular value of Γ, i.e., the largest eigenvalue of $\Gamma^T\Gamma$ or $\Gamma\Gamma^T$.

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Theorem (Albert and V). *Let* $\mathcal{M}$ *be a* $t \times u$ *matrix of permutation classes, each with a proper growth rate, and define the matrix* Γ *of the same size by*

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The growth rate of $\mathrm{Grid}(\mathcal{M})$ *is equal to the greatest eigenvalue of* $\Gamma^T\Gamma$ *(or equivalently, of* $\Gamma\Gamma^T$*).*

The Lower Bound on $\mathrm{gr}(\mathrm{Av}(k\cdots 21))$

$\mathrm{Av}(k\cdots 21)$ contains

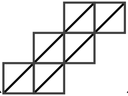
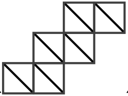
$$\mathrm{Grid}\begin{pmatrix} & & & \iddots & \iddots \\ & & \mathrm{Av}(21) & \mathrm{Av}((k-1)\cdots 21) & \\ \mathrm{Av}(21) & \mathrm{Av}((k-1)\cdots 21) & & & \end{pmatrix}.$$

(The $\mathrm{Av}(21)$ cells may be taken to contain the left-to-right maxima.)

$$\Gamma = \begin{pmatrix} & & & \iddots & \iddots \\ & & 1 & k-2 & \\ 1 & k-2 & & & \end{pmatrix}.$$

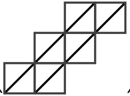
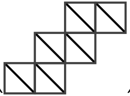
The largest eigenvalues of $\Gamma\Gamma^T$ tend to $(k-1)^2$ (as we let Γ grow).

A Mystery

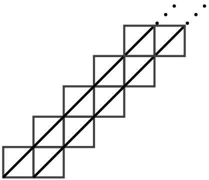
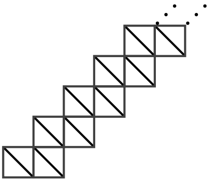
$\mathrm{gr}\Bigg($  $\Bigg) = \mathrm{gr}\Bigg($  $\Bigg)$

For *all* numbers of cells.

A Mystery

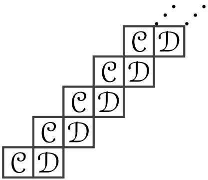
gr() = gr()

For *all* numbers of cells. Yet...

$4 = \mathrm{gr}$() $< \mathrm{gr}$() $\approx 4.5189.$

(Approximation of 4.5189 due to Jay / differential approximates.)

The General $(\mathcal{C}, \mathcal{D})$ Staircase



Assuming $\mathcal{C}$ and $\mathcal{D}$ have proper growth rates, $\Gamma\Gamma^T$ is the tridiagonal Toeplitz matrix defined by

$$(\Gamma\Gamma^T)_{k,\ell} = \begin{cases} \mathrm{gr}(\mathcal{C}) + \mathrm{gr}(\mathcal{D}) & \text{if } k = \ell, \\ \sqrt{\mathrm{gr}(\mathcal{C})\,\mathrm{gr}(\mathcal{D})} & \text{if } |k - \ell| = 1, \text{ and} \\ 0 & \text{otherwise.} \end{cases}$$

A linear algebra result shows that the growth rate of the $(\mathcal{C}, \mathcal{D})$ staircase is equal to

$$(\sqrt{\mathrm{gr}(\mathcal{C})} + \sqrt{\mathrm{gr}(\mathcal{D})})^2.$$

$\mathcal{C} \odot \mathcal{D}$

The merge of $\mathcal{C}$ and $\mathcal{D}$: all permutations whose entries can be colored red and blue so that the red subsequence is order isomorphic to a member of $\mathcal{C}$ and the blue subsequence is order isomorphic to a member of $\mathcal{D}$.

$\mathcal{C} \odot \mathcal{D}$

The merge of $\mathcal{C}$ and $\mathcal{D}$: all permutations whose entries can be colored red and blue so that the red subsequence is order isomorphic to a member of $\mathcal{C}$ and the blue subsequence is order isomorphic to a member of $\mathcal{D}$.
Obvious bound:

$$\begin{aligned} \mathrm{gr}(\mathcal{C} \odot \mathcal{D}) &\leqslant \sum_{i=0}^{n} \binom{n}{i}^2 |\mathcal{C}_i||\mathcal{D}_{n-i}|, \\ &\leqslant \left(\sum_{i=0}^{n} \binom{n}{i} \sqrt{|\mathcal{C}_i||\mathcal{D}_{n-i}|} \right)^2. \end{aligned}$$

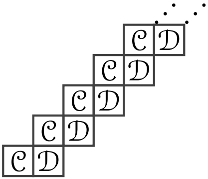
Using the Binomial Theorem (assuming all growth rates below exist):

$$\mathrm{gr}(\mathcal{C} \odot \mathcal{D}) \leqslant \left(\sqrt{\mathrm{gr}(\mathcal{C})} + \sqrt{\mathrm{gr}(\mathcal{D})} \right)^2.$$

When is This the Answer?

If both $\mathcal{C}$ and $\mathcal{D}$ are sum closed then:

- Both $\mathrm{gr}(\mathcal{C})$ and $\mathrm{gr}(\mathcal{D})$ exist.
- $\mathcal{C} \odot \mathcal{D}$ contains the $(\mathcal{C}, \mathcal{D})$ staircase:



Corollary. *If both $\mathcal{C}$ and $\mathcal{D}$ are sum closed (or by symmetry, both are skew closed) then $\mathrm{gr}(\mathcal{C} \odot \mathcal{D})$ exists and equals*

$$\left(\sqrt{\mathrm{gr}(\mathcal{C})} + \sqrt{\mathrm{gr}(\mathcal{D})}\right)^2.$$

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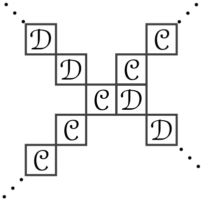
If $\mathcal{C}$ is sum closed and $\mathcal{D}$ is skew closed then:

- Both $\mathrm{gr}(\mathcal{C})$ and $\mathrm{gr}(\mathcal{D})$ exist.
- $\mathcal{C} \odot \mathcal{D}$ contains the $(\mathcal{C}, \mathcal{D})$ spiral staircase:



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$$\left(\sqrt{\mathrm{gr}(\mathcal{C})} + \sqrt{\mathrm{gr}(\mathcal{D})}\right)^2.$$

When is This the Answer?

Recall: Classes defined by avoiding a single pattern, the *principal classes*, are always either sum closed or skew closed.

Corollary. *For all permutations* β *and* γ, $\mathrm{gr}(\mathrm{Av}(\beta) \odot \mathrm{Av}(\gamma))$ *exists and equals*

$$\left(\sqrt{\mathrm{gr}(\mathrm{Av}(\beta))} + \sqrt{\mathrm{gr}(\mathrm{Av}(\gamma))}\right)^2.$$

Conclusion

Question. *Does*

$$\mathrm{gr}(\mathcal{C}) = \lim_{n\to\infty} \sqrt[n]{|\mathcal{C}_n|}$$

exist for all proper permutation classes $\mathcal{C}$*?*

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for all permutation classes $\mathcal{C}$ *and* $\mathcal{D}$*?*

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Thank you.