Yangians, quantum loop algebras and elliptic quantum groups

(joint with Sachin Gautam)

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February 10, 2016

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- All the results are valid for a symmetrisable Kac–Moody algebra $\mathfrak{g}$.
- For notational simplicity, restrict attention to $\mathfrak{g} = \mathfrak{sl}_2 = \langle e, f, h \rangle$.

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Relations for any $r, s \in \mathbb{N}$

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$$[\xi(u), \xi(v)] = 0$$

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Prop (GTL) On $V \in \mathrm{Rep}_{\mathrm{fd}}(Y_\hbar(\mathfrak{g}))$, the fields $\xi(u)$, $x^\pm(u)$ are the Taylor expansions at $u = \infty$ of $\mathrm{End}(V)$–valued rational functions.

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$$\mathcal{X}^\pm(z)\mathcal{X}^\pm(w) = \frac{q^{\pm 2}z - w}{z - q^{\pm 2}w}\mathcal{X}^\pm(w)\mathcal{X}^\pm(z) \mp \frac{1 - q^{\pm 2}}{z - q^{\pm 2}w}(w\mathcal{X}^\pm(z)^2 + z\mathcal{X}^\pm(w)^2$$

$$[\mathcal{X}^+(z), \mathcal{X}^-(w)] = \frac{1}{q - q^{-1}}\left(\frac{z\Psi(w) - w\Psi(z)}{z - w} - \Psi(0)\right)$$

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Problem Construct a functor $\mathcal{F} : \mathrm{Rep}_{\mathrm{fd}}(Y_\hbar(\mathfrak{g})) \to \mathrm{Rep}_{\mathrm{fd}}(U_q(L\mathfrak{g}))$ which induces $\mathcal{E}\mathrm{xp}$,

Problem # 1: relation between $Y_\hbar(\mathfrak{g})$ and $U_q(L\mathfrak{g})$?

"$Y_\hbar(\mathfrak{g})$ and $U_q(L\mathfrak{g})$ have the same f.d. representation theory"

$$\begin{array}{ccc} \mathrm{Irrep}_{\mathrm{fd}}(Y_\hbar(\mathfrak{g})) & \longleftrightarrow & \bigcup_{n\geq 0} \mathbb{C}^n/\mathfrak{S}_n \\ \downarrow \mathcal{E}\mathrm{xp} & & \downarrow \exp(2\pi\iota -) \\ \mathrm{Irrep}_{\mathrm{fd}}(U_q(L\mathfrak{g})) & \longleftrightarrow & \bigcup_{n\geq 0} (\mathbb{C}^\times)^n/\mathfrak{S}_n \end{array}$$

Theorem (Nakajima, Varagnolo) If $\mathfrak{g}$ is simply–laced, $\mathcal{E}\mathrm{xp}$ preserves dimensions.

Caveat $\mathcal{E}\mathrm{xp}$ is a set–theoretic map, **not** a functor.

Problem Construct a functor $\mathcal{F} : \mathrm{Rep}_{\mathrm{fd}}(Y_\hbar(\mathfrak{g})) \to \mathrm{Rep}_{\mathrm{fd}}(U_q(L\mathfrak{g}))$ which induces $\mathcal{E}\mathrm{xp}$, and an equivalence of appropriate subcategories.

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Main ingredient Γ is governed by an abelian difference equation.

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Remark. $S(u)$ is a regularisation of

$$\cdots A(u+2)A(u+1)A(u)A(u-1)A(u-2)\cdots$$

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Main idea

- Recall that $\xi(u) \in GL(V)(u)$, $\xi(\infty) = 1$
- Consider the additive difference equation

$$f(u+1) = \xi(u) f(u)$$

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$$\Psi(z) \longrightarrow S(z) = \cdots \xi(u+1)\xi(u)\xi(u-1)\cdots$$

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Remark The inverse functor is governed by the Riemann–Hilbert problem $S(z) \rightsquigarrow A(u)$ (always solvable since $[S(z), S(w)] = 0$).

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Remark (2) is a meromorphic, q–deformed version of the Kazhdan–Lusztig equivalence $\mathcal{O}_\kappa(\widehat{\mathfrak{g}}) \xrightarrow{\sim} \mathrm{Rep}_{\mathrm{fd}}(U_q\mathfrak{g})$.

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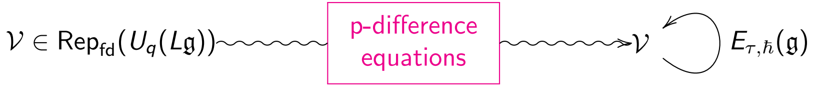

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2. These relations and the quasi–periodicity properties were already worked out by Enriquez–Felder (1998), in connection with a Drinfeld–type presentation of Felder's elliptic quantum group $E_{\tau,\hbar}(\mathfrak{sl}_2)$.

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\varphi(\lambda)h &= h\varphi(\lambda)\\
\varphi(\lambda)\Phi(u) &= \Phi(u)\varphi(\lambda + 2\hbar)\\
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\varphi(\lambda - \hbar h - \hbar)\mathfrak{X}^{-}(u,\lambda) &= \mathfrak{X}^{-}(u,\lambda)\varphi(\lambda - \hbar h + \hbar)
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