Discretizing the Advection of Differential Forms: Semi-Lagrangian Techniques

Ralf Hiptmair[1], Holger Heumann[2]

[1] Seminar for Applied Mathematics, ETH Zürich
[2] Laboratoire J.A.Dieudonné, UMR CNRS, Université de Nice Sophia-Antipolis

Computational and Numerical Analysis
of Transient Problems
in Acoustics, Elasticity, and Electromagnetism
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material laws

$$\begin{aligned} \mathbf{B} &= \mu \mathbf{H}\,, \\ \text{Ohm:} \quad \mathbf{j} &= \sigma(\mathbf{E} + \mathbf{v} \times \mathbf{B})\,. \end{aligned}$$

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Focus: $\epsilon^{-1} := R_m := \|\mathbf{v}\|\, \mu\sigma\, \mathrm{diam}(\Omega) \gg 1$ ➥ transport dominates

What Next ?

1. Generalized Advection-Diffusion Problems
2. Semi-Lagrangian Timestepping
3. Semi-Lagrangian Scheme: Pure Advection
4. SL for Advection-Diffusion: Convergence

Boundary Value Problems in Exterior Calculus

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Recall: general 2nd-order "diffusion" boundary value problem

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\begin{aligned}
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The **guiding principle**:

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A (numerical) method works well for BVPs for 0-forms

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► Try it/adapt it to BVPs for ℓ-forms

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$\{\mathbf{\Phi}_t : \Omega \mapsto \Omega\}_t \quad \hat{=} \quad$ flow map induced by velocity $\mathbf{v} = \mathbf{v}(\boldsymbol{x}) \quad (\mathbf{v} \cdot \boldsymbol{n}_{|\partial\Omega} = 0)$

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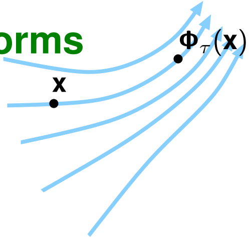

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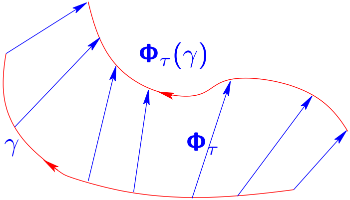

$\gamma \;\hat{=}\;$ integration path

(1-form ω assigns a number to each oriented integration path.)

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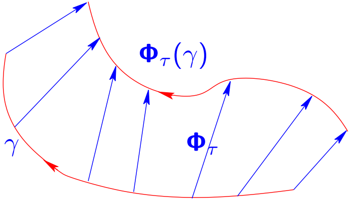

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Rate of change during transport: Material derivative

$$\int_M (\mathrm{D_t}\,\omega)(t) := \frac{d}{d\tau} \int_{\mathbf{\Phi}_\tau(M)} \omega \bigg|_{\tau=t} =$$

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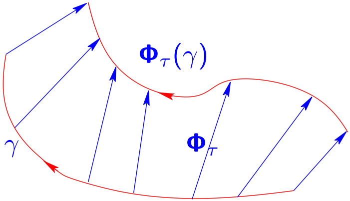
$\gamma \hat{=}$ integration path

(1-form ω assigns a number to each oriented integration path.)



Rate of change during transport: Material derivative

$$\int_M (\mathsf{D}_t\,\omega)(t) := \frac{d}{d\tau} \int_{\boldsymbol{\Phi}_\tau(M)} \omega \Big|_{\tau=t} = \lim_{\tau \to 0} \frac{1}{\tau} \left[\int_M \omega(t) - \int_{\boldsymbol{\Phi}_{-\tau}(M)} \omega(t-\tau) \right] .$$

Transport Operators

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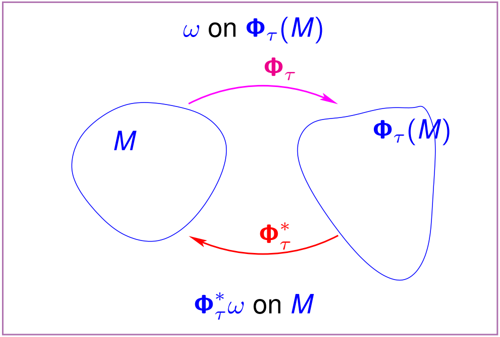

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Local material derivative

For time-dependent differential form $\omega = \omega(t)$:

$$\mathsf{D}_t\,\omega(t) =$$

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"transport theorem"

$$\mathsf{D}_t\,\omega(t) = \frac{d}{d\tau}\mathbf{\Phi}_\tau^*\omega(t)\bigg|_{\tau=t} = \partial_t\omega + \mathcal{L}_{\mathbf{v}}\omega .$$

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Generalized Advection-Diffusion Problems

Generalized ADP for ℓ-forms:

$$\star_\sigma(\partial_t\omega + \mathcal{L}_{\mathbf{v}}\omega) + (-1)^{\ell-1}\,\mathsf{d}\star_\alpha\mathsf{d}\omega = \varphi,$$

$$\mathbf{t}_\partial\,\omega = 0 \quad \text{on } \partial\Omega.$$

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$$\ell = 2: \quad \begin{cases} \partial_t \mathbf{u} + \mathbf{curl}(\mathbf{u}\times\mathbf{v}) + \operatorname{div}\mathbf{u}\cdot\mathbf{v} - \mathbf{grad}(\alpha\operatorname{div}\mathbf{u}) &= \mathbf{f} \quad \text{in } \Omega\,, \\ \mathbf{u}(t)\cdot\boldsymbol{n} &= 0 \quad \text{on } \partial\Omega\,, \end{cases}$$

Generalized Advection-Diffusion Problems

Cartan's "magic formula": $\boxed{\mathcal{L}_{\mathbf{v}}\,\omega = \mathrm{d}(\imath_{\mathbf{v}}\omega) + \imath_{\mathbf{v}}(\mathrm{d}\,\omega)}$.

Generalized ADP for ℓ-forms:
$$\star_\sigma(\partial_t\omega + \mathcal{L}_{\mathbf{v}}\omega) + (-1)^{\ell-1}\,\mathrm{d}\star_\alpha \mathrm{d}\omega = \varphi,$$
$$\mathbf{t}_\partial\,\omega = 0 \quad \text{on } \partial\Omega.$$

Vector proxy incarnation in 3D ($\sigma \equiv 1$):

$$\ell = 0:\quad \begin{cases} \partial_t u + \mathbf{v}\cdot\mathbf{grad}\,u - \operatorname{div}(\alpha\,\mathbf{grad}\,u) &= f \quad \text{in } \Omega\,, \\ u(t) &= 0 \quad \text{on } \partial\Omega\,, \end{cases}$$

$$\ell = 1:\quad \begin{cases} \partial_t\mathbf{u} + \mathbf{grad}(\mathbf{u}\cdot\mathbf{v}) + \mathbf{curl}\,\mathbf{u}\times\mathbf{v} + \mathbf{curl}(\alpha\,\mathbf{curl}\,\mathbf{u}) &= \mathbf{f} \quad \text{in } \Omega\,, \\ \mathbf{u}(t)\times\boldsymbol{n} &= 0 \quad \text{on } \partial\Omega\,, \end{cases}$$

$$\ell = 2:\quad \begin{cases} \partial_t\mathbf{u} + \mathbf{curl}(\mathbf{u}\times\mathbf{v}) + \operatorname{div}\mathbf{u}\cdot\mathbf{v} - \mathbf{grad}(\alpha\,\operatorname{div}\mathbf{u}) &= \mathbf{f} \quad \text{in } \Omega\,, \\ \mathbf{u}(t)\cdot\boldsymbol{n} &= 0 \quad \text{on } \partial\Omega\,, \end{cases}$$

$$\ell = 3:\quad \left\{\ \partial_t u + \operatorname{div}(u\cdot\mathbf{v}) = f \quad \text{in } \Omega\,. \right.$$

Generalized Advection-Diffusion Problems

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$$\ell = 2: \quad \left\{ \begin{array}{rcll} \partial_t\mathbf{u} + \mathbf{curl}(\mathbf{u}\times\mathbf{v}) + \operatorname{div}\mathbf{u}\cdot\mathbf{v} - \mathbf{grad}(\alpha\operatorname{div}\mathbf{u}) & = & \mathbf{f} & \text{in } \Omega\,, \\ \mathbf{u}(t)\cdot\boldsymbol{n} & = & 0 & \text{on } \partial\Omega\,, \end{array} \right.$$

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Generalized Advection-Diffusion Problems

Cartan's "magic formula": $\mathcal{L}_{\mathbf{v}}\omega := \mathsf{d}(\mathsf{i}_{\mathbf{v}}\omega) + \mathsf{i}_{\mathbf{v}}(\mathsf{d}\omega)$

Recall: eddy currents in moving conductors

$$\sigma\partial_t \mathbf{A} + \sigma\,\mathbf{curl}\,\mathbf{A}\times\mathbf{v} + \mathbf{curl}(\mu^{-1}\,\mathbf{curl}\,\mathbf{A}) = \mathbf{j}_s\ .$$

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Vector proxy incarnation in 3D ($\partial = 1$):

$$\ell = 0:\quad \begin{cases} \partial_t u + \mathbf{v}\cdot\mathbf{grad}\,u - \operatorname{div}(\alpha\,\mathbf{grad}\,u) = f & \text{in } \Omega\ ,\\ u(t) = 0 & \text{on } \partial\Omega\ , \end{cases}$$

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Magnetic Advection-Diffusion

Magneto-quasistatic model, conducting fluid moving with velocity $\mathbf{v}$:

$$\begin{aligned} \mathbf{curl}\,\mathbf{E} &= -\partial_t \mathbf{B}\,, & \mathbf{j} &= \sigma(\mathbf{E} + \mathbf{v} \times \mathbf{B})\,, \\ \mathbf{curl}\,\mathbf{H} &= \mathbf{j} + \mathbf{j}_s\,, & \mathbf{B} &= \mu \mathbf{H}\,. \end{aligned}$$

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transformation: $\widetilde{\mathbf{E}} := \mathbf{E} + \mathbf{v} \times \mathbf{B}$

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In the language of differential forms:

$$\begin{aligned} \widetilde{\mathbf{E}} &\leftrightarrow \text{1-form } \mathbf{e} \\ \mathbf{B} &\leftrightarrow \text{2-form } \mathbf{b} \\ \mathbf{H} &\leftrightarrow \text{1-form } \mathbf{h} \\ \mathbf{j} &\leftrightarrow \text{2-form } \mathbf{j} \end{aligned}$$

Magnetic Advection-Diffusion

Magneto-quasistatic model, conducting fluid moving with velocity $\mathbf{v}$:

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In the language of differential forms:

$\widetilde{\mathbf{E}}$	$\leftrightarrow$	1-form $\mathbf{e}$
$\mathbf{B}$	$\leftrightarrow$	2-form $\mathbf{b}$
$\mathbf{H}$	$\leftrightarrow$	1-form $\mathbf{h}$
$\mathbf{j}$	$\leftrightarrow$	2-form $\mathbf{j}$

$$\boxed{\mathsf{d}\,\mathbf{e} = -\partial_t \mathbf{b} - \mathsf{d}(\imath_{\mathbf{v}} \mathbf{b}) \quad ,}$$

Magnetic Advection-Diffusion

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Magnetic Advection-Diffusion

Magneto-quasistatic model, conducting fluid moving with velocity $\mathbf{v}$:

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$$\boxed{\begin{aligned} \mathrm{d}\,\mathbf{e} &= -\partial_t \mathbf{b} - \mathrm{d}(\imath_{\mathbf{v}}\,\mathbf{b}) \\ \mathrm{d}\,\mathbf{h} &= \mathbf{j}\,, \\ \mathbf{j} &= \star_\sigma \mathbf{e}\,, \quad \mathbf{b} = \star_\mu \mathbf{h}\,. \end{aligned}}$$

Magnetic Advection-Diffusion

Magneto-quasistatic model, conducting fluid moving with velocity $\mathbf{v}$:

$$\begin{aligned} \mathbf{curl}\,\mathbf{E} &= -\partial_t \mathbf{B}\,, & \mathbf{j} &= \sigma(\mathbf{E} + \mathbf{v}\times\mathbf{B})\,,\\ \mathbf{curl}\,\mathbf{H} &= \mathbf{j} + \mathbf{j}_s\,, & \mathbf{B} &= \mu\mathbf{H}\,. \end{aligned}$$

transformation: $\widetilde{\mathbf{E}} := \mathbf{E} + \mathbf{v}\times\mathbf{B}$

$$\begin{aligned} \mathbf{curl}\,\widetilde{\mathbf{E}} &= -\partial_t\mathbf{B} - \mathbf{curl}(\mathbf{v}\times\mathbf{B}) - \underbrace{\mathbf{v}\,\mathrm{div}\,\mathbf{B}}_{=0}\,, & \mathbf{j} &= \sigma\widetilde{\mathbf{E}}\,,\\ \mathbf{curl}\,\mathbf{H} &= \mathbf{j} + \mathbf{j}_s\,, & \mathbf{B} &= \mu\mathbf{H}\,. \end{aligned}$$

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Magnetic Advection-Diffusion

Magneto-quasistatic model, conducting fluid moving with velocity $\mathbf{v}$:

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transformation: $\widetilde{\mathbf{E}} := \mathbf{E} + \mathbf{v} \times \mathbf{B}$

▶

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$$\boxed{\begin{aligned} \mathrm{d}\,\mathbf{e} &= -\partial_t \mathbf{b} - \mathrm{d}(\imath_{\mathbf{v}}\,\mathbf{b}) - \imath_{\mathbf{v}}\underbrace{(\mathrm{d}\,\mathbf{b})}_{=0} = -\mathsf{D}_t\,\mathbf{b} \,, \\ \mathrm{d}\,\mathbf{h} &= \mathbf{j} \,, \\ \mathbf{j} &= \star_\sigma \mathbf{e} \,, \quad \mathbf{b} = \star_\mu \mathbf{h} \,. \end{aligned}}$$

Magnetic Advection-Diffusion IBVP

$$\boxed{\begin{array}{c} \begin{array}{rclcrcl} \mathsf{d}\,\mathbf{e} & = & -\mathsf{D_t}\,\mathbf{b} & , & \mathsf{d}\,\mathbf{h} & = & \mathbf{j}+\mathbf{j}_0\,, \\ \mathbf{j} & = & \star_\sigma \mathbf{e} & , & \mathbf{b} & = & \star_\mu \mathbf{h} \end{array} \quad \text{in } \Omega\,, \\ \mathbf{t}_\partial\,\mathbf{e} = 0 \quad \text{on } \partial\Omega\,. \end{array}}$$

Magnetic Advection-Diffusion IBVP

$$\boxed{\begin{array}{rclcrcl} \mathsf{d}\,\mathbf{e} & = & -\mathsf{D}_t\,\mathbf{b} & , & \mathsf{d}\,\mathbf{h} & = & \mathbf{j}+\mathbf{j}_0\,, \\ \mathbf{j} & = & \star_\sigma \mathbf{e} & , & \mathbf{b} & = & \star_\mu \mathbf{h} \end{array} \quad \text{in } \Omega\,, \\ \mathbf{t}_\partial\,\mathbf{e} = 0 \quad \text{on } \partial\Omega\,.}$$

Magnetic vector potential:

$$\mathbf{a} \in \mathcal{F}^1: \quad \mathsf{d}\,\mathbf{a} = \mathbf{b} \quad \Rightarrow \quad \mathbf{e} = -\,\mathsf{D}_t\,\mathbf{a} \quad \text{(advected temporal gauge)}\,.$$

Magnetic Advection-Diffusion IBVP

$$\begin{array}{rclcrcl} \mathsf{d}\,\mathbf{e} & = & -\mathsf{D_t}\,\mathbf{b} & , & \mathsf{d}\,\mathbf{h} & = & \mathbf{j} + \mathbf{j}_0\,, \\ \mathbf{j} & = & \star_\sigma \mathbf{e} & , & \mathbf{b} & = & \star_\mu \mathbf{h} \end{array} \quad \text{in } \Omega\,,$$

$$\mathbf{t}_\partial\,\mathbf{e} = 0 \quad \text{on } \partial\Omega\,.$$

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$$\star_\sigma \mathsf{D_t}\,\mathbf{a} + \mathsf{d}(\star_\mu^{-1}\,\mathsf{d}\,\mathbf{a}) = \mathbf{j}_0 \quad \text{in } \Omega\,,$$

$$\mathbf{t}_\partial\,\mathbf{a} = 0 \quad \text{on } \partial\Omega \quad , \quad \mathbf{a}(0) = \mathbf{a}_0\,.$$

Magnetic Advection-Diffusion IBVP

$$\boxed{\begin{array}{c} \begin{array}{rclcrcl} \mathrm{d}\,\mathbf{e} & = & -\mathsf{D}_t\,\mathbf{b} & , & \mathrm{d}\,\mathbf{h} & = & \mathbf{j}+\mathbf{j}_0\,, \\ \mathbf{j} & = & \star_\sigma \mathbf{e} & , & \mathbf{b} & = & \star_\mu \mathbf{h} \end{array} \quad \text{in } \Omega\,, \\ \mathbf{t}_\partial\,\mathbf{e} = 0 \quad \text{on } \partial\Omega\,. \end{array}}$$

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$$\boxed{\begin{array}{c} \star_\sigma \mathsf{D}_t\,\mathbf{a} + \mathrm{d}(\star_\mu^{-1}\,\mathrm{d}\,\mathbf{a}) = \mathbf{j}_0 \quad \text{in } \Omega\,, \\ \mathbf{t}_\partial\,\mathbf{a} = 0 \quad \text{on } \partial\Omega \quad , \quad \mathbf{a}(0) = \mathbf{a}_0\,. \\ \Updownarrow \\ \sigma(\partial_t \mathbf{A} + \mathbf{curl}\,\mathbf{A} \times \mathbf{v} + \mathbf{grad}(\mathbf{A}\cdot\mathbf{v})) + \mathbf{curl}(\mu^{-1}\,\mathbf{curl}\,\mathbf{A}) = \mathbf{j}_0 \quad \text{in } \Omega\,, \\ \mathbf{A} \times \mathbf{n} = 0 \quad \text{on } \partial\Omega \quad , \quad \mathbf{a}(0) = \mathbf{A}_0\,. \end{array}}$$

Magnetic Advection-Diffusion IBVP

$$\boxed{\begin{array}{rclcrcll} \mathsf{d}\,\mathbf{e} &=& -\mathsf{D}_t\,\mathbf{b} &,& \mathsf{d}\,\mathbf{h} &=& \mathbf{j}+\mathbf{j}_0\,, & \\ \mathbf{j} &=& \star_\sigma \mathbf{e} &,& \mathbf{b} &=& \star_\mu \mathbf{h} & \text{in } \Omega\,, \\ \multicolumn{7}{c}{\mathbf{t}_\partial\,\mathbf{e} = 0 \quad \text{on } \partial\Omega\,.} \end{array}}$$

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$$\mathbf{t}_\partial\,\mathbf{a} = 0 \quad \text{on } \partial\Omega \quad , \quad \mathbf{a}(0) = \mathbf{a}_0\,.$$
$$\Updownarrow$$
$$\sigma(\partial_t \mathbf{A} + \mathbf{curl}\,\mathbf{A} \times \mathbf{v} + \mathbf{grad}(\mathbf{A}\cdot\mathbf{v})) + \mathbf{curl}(\mu^{-1}\,\mathbf{curl}\,\mathbf{A}) = \mathbf{j}_0 \quad \text{in } \Omega\,,$$
$$\mathbf{A} \times \mathbf{n} = 0 \quad \text{on } \partial\Omega \quad , \quad \mathbf{a}(0) = \mathbf{A}_0\,.$$

Magnetic Advection-Diffusion IBVP

$$\boxed{\begin{array}{rclcrcl} \mathrm{d}\,\mathbf{e} &=& -\mathrm{D_t}\,\mathbf{b} &,& \mathrm{d}\,\mathbf{h} &=& \mathbf{j}+\mathbf{j}_0\;, \\ \mathbf{j} &=& \star_\sigma \mathbf{e} &,& \mathbf{b} &=& \star_\mu \mathbf{h} \end{array} \quad \text{in } \Omega\;, \\ \mathbf{t}_\partial\,\mathbf{e} = 0 \quad \text{on } \partial\Omega\;.}$$

Magnetic vector potential:

$$\mathbf{a} \in \mathcal{F}^1: \quad \mathrm{d}\,\mathbf{a} = \mathbf{b} \quad \Rightarrow \quad \mathbf{e} = -\,\mathrm{D_t}\,\mathbf{a} \quad \text{(advected temporal gauge)}\;.$$

$$\star_\sigma \mathrm{D_t}\,\mathbf{a} + \mathrm{d}(\star_\mu^{-1}\,\mathrm{d}\,\mathbf{a}) = \mathbf{j}_0 \quad \text{in } \Omega\;,$$

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$$\Updownarrow$$

$$\sigma(\partial_t \mathbf{A} + \mathbf{curl}\,\mathbf{A} \times \mathbf{v} + \mathbf{grad}(\mathbf{A}\cdot\mathbf{v})) + \mathbf{curl}(\mu^{-1}\,\mathbf{curl}\,\mathbf{A}) = \mathbf{j}_0 \quad \text{in } \Omega\;,$$

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What Next ?

Weak Advection-Diffusion BVP

Singularly perturbed ($\epsilon \ll 1$) BVP for ℓ-form $\omega = \omega(t) \in \Lambda^\ell(\Omega)$:

$$\star \mathsf{D}_t \omega + \epsilon (-1)^{\ell-1} \mathsf{d}(\star \mathsf{d}\, \omega) = \varphi(t) \quad \text{in } \Omega \, ,$$
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Realization of $\mathsf{P}_h \mathbf{\Phi}_{-\tau}^*$ is **the** pivotal issue in SL schemes.

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- ϵ-Robust estimates for advection-diffusion:

  Bause & Knabner (2002), Wang & Wang (2010), Bermejo & Saavedra (2012), all $O(\tau + h^2 + h^2/\tau)$

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What Next ?

Pure Advection: Convergence

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Needed: $\left\{ \begin{array}{l} \\ P_h\mathbf{\Phi}^*_{-\tau}\omega_h \text{ can be computed exactly,} \end{array} \right.$

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\end{aligned}$$

Needed: $$\begin{cases} \left\|\mathbf{\Phi}^*_{-\tau}\omega\right\|^2_{L^2} \leq (1 + C\tau)\,\|\omega\|^2_{L^2}\,. \\ P_h\mathbf{\Phi}^*_{-\tau}\omega_h \text{ can be computed exactly,} \end{cases}$$

Pure Advection: Convergence

$$\mathsf{D}_t\,\omega = \varphi \quad \succ \quad \omega(t^n) = \mathbf{\Phi}^*_{-\tau}\omega(t^{n-1}) + \int_{t^{n-1}}^{t^n} \mathbf{\Phi}^*_{\tau-t^n}\varphi(\tau)\mathrm{d}\tau$$

SL: $(\omega_h^n \in \Lambda_h^k(\mathcal{T}))$ $\quad \succ \quad \omega_h^n = P_h\mathbf{\Phi}^*_{-\tau}\omega_h^{n-1} + P_h\int_{t^{n-1}}^{t^n} \mathbf{\Phi}^*_{\tau-t^n}\varphi(\tau)\mathrm{d}\tau$

Theorem Idea: P_h is L^2-projection $\succ$ use Pythagoras:

$$\begin{aligned}
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$$\overset{\text{Gronwall}}{\Rightarrow} \max_{0\leq n\leq N} \|\omega(t^n) - \omega_h^n\|_{L^2} \leq C\tau^{-\frac{1}{2}} \max_{0\leq n\leq N} \|\omega(t^n) - P_h\omega(t^n)\|_{L^2} = O(\tau^{-\frac{1}{2}}h^{r+1}).$$

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Pure Advection: Empiric Convergence

Numerical Experiment: $\ell = 0$, scalar advection, monitor L^2-error
rotating bump on unit-circle, $\mathbf{v} = (-y, x)$, smooth initial data, $\tau = \frac{0.8}{\sqrt{2}}h$

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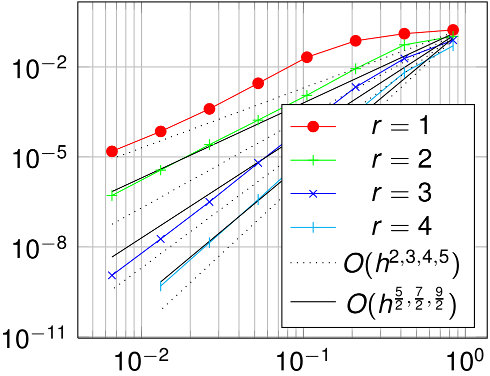

continuous finite elements

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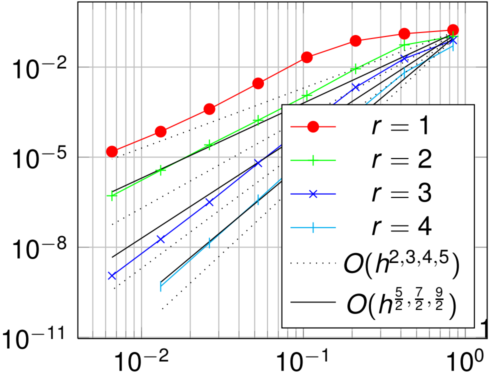

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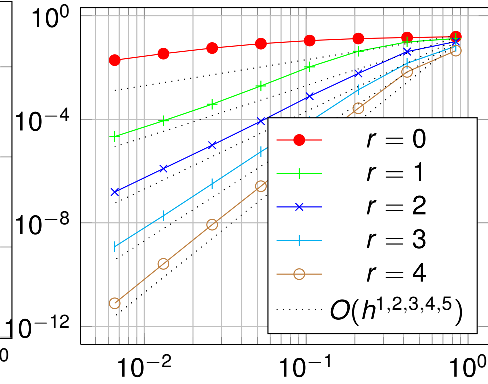

discontinuous finite elements

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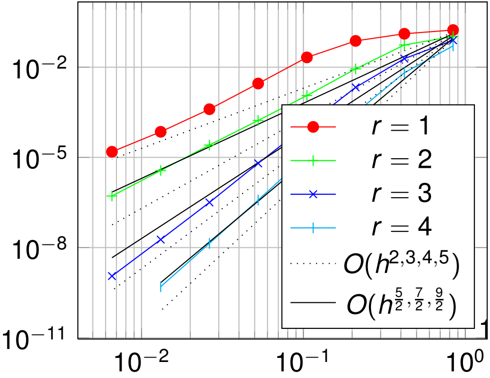

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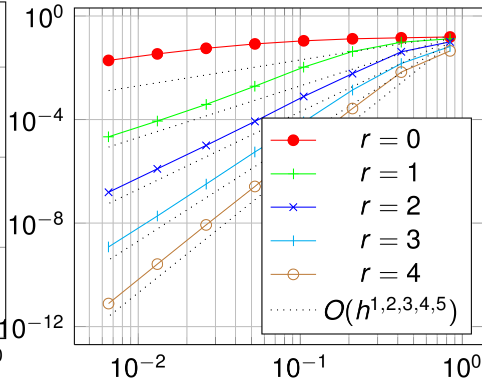

discontinuous finite elements

Super-convergence $O(h^{r+1})$(vs. $O(h^{-\frac{1}{2}} h^{r+1})$) except for cont. elements, r even.

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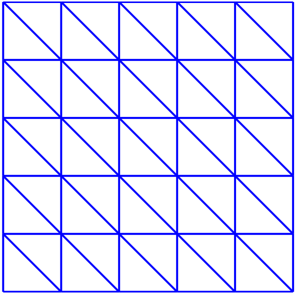

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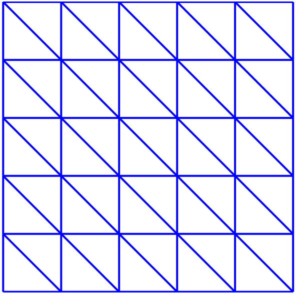
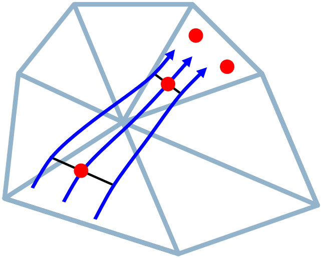


- FEM-Quadrature:

Dangerous Quadrature

Numerical Experiment: Pure advection ($\ell = 1$, $d = 2$) and quadrature

$$\left(\mathbf{\Phi}^*_{-\tau}\omega_h, \eta_h\right)_\Omega \approx \sum_{K \in \mathcal{T}} \mathbf{\Phi}^*_{-\tau}\omega_h \eta_{h K,h} \quad \omega_h \in \Lambda^1_h(\mathcal{T}), \text{ lowest order}.$$



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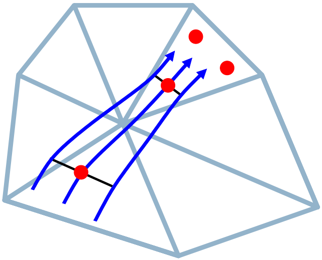

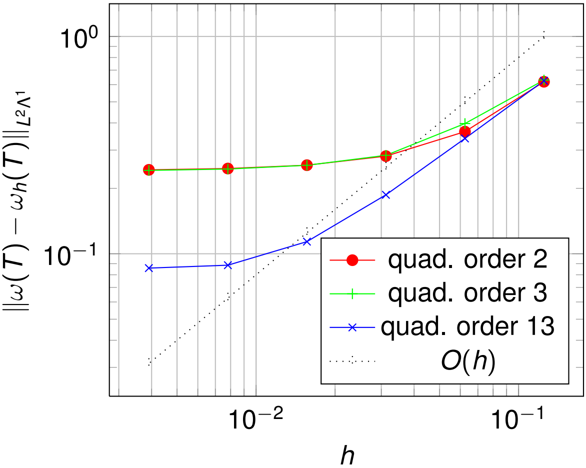

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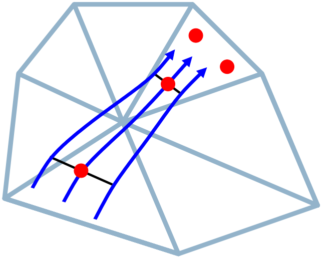
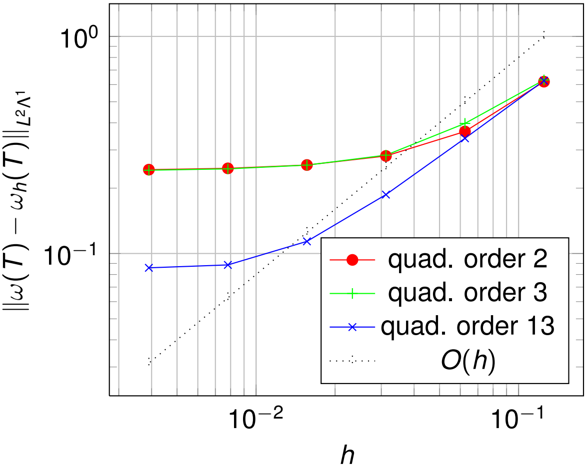


No convergence, why?



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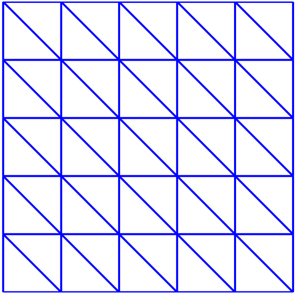

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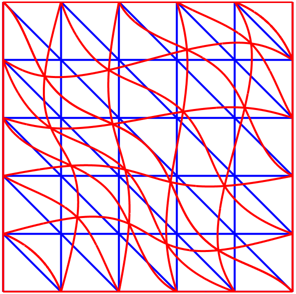

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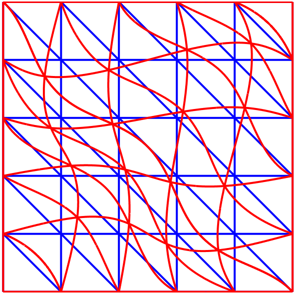

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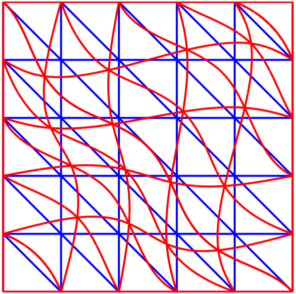

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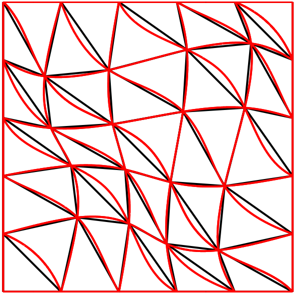

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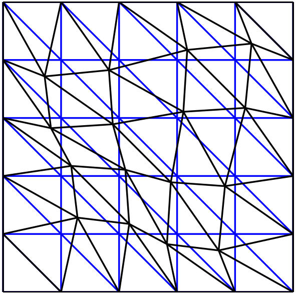

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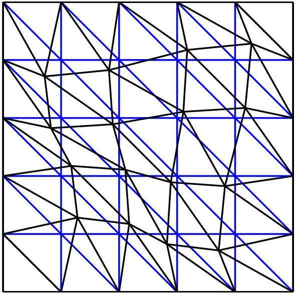

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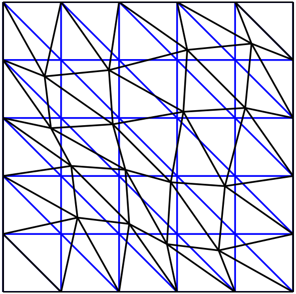

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In SL scheme: replace $\mathsf{P}_h \longrightarrow$ nodal interpolation I_h

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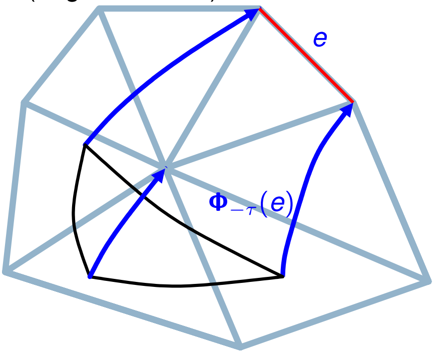

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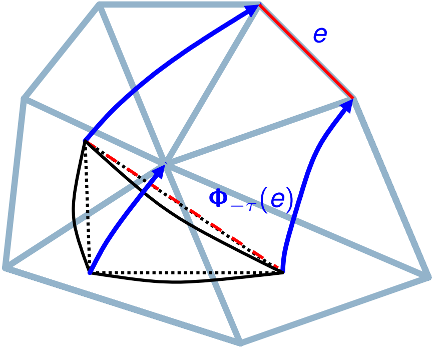

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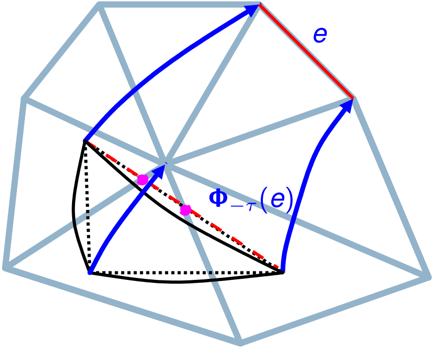
▷ straight approximation of $\boldsymbol{\Phi}_{-\tau}(e)$

▷ detect intersection with faces of $\mathcal{T}$



Semi-Lagrangian Interpolation Scheme (I)

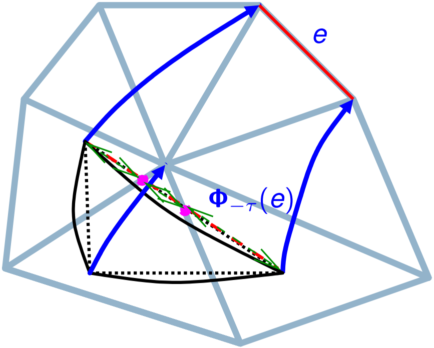
In SL scheme: replace $\mathsf{P}_h \longrightarrow$ nodal interpolation I_h

$$\left(\frac{\omega_h^n - \mathsf{I}_h \boldsymbol{\Phi}_{-\tau}^* \omega_h^{n-1}}{\tau}, \omega_h'\right)_\Omega + \epsilon\,(\mathsf{d}\,\omega_h^n, \mathsf{d}\,\omega_h')_\Omega = \frac{1}{\tau}\int_{t_{n-1}}^{t_n} (\varphi(\xi), \omega_h')_\Omega \,\mathrm{d}\xi \quad \forall \omega_h' \in \Lambda_h^\ell(\mathcal{T})\,.$$

Example: $\ell = 1$, lowest order FE space (edge elements)

Task: evaluate $\int_{\boldsymbol{\Phi}_{-\tau}(e)} \beta_j$ for all mesh edges e, basis 1-forms β_j

- ▷ straight approximation of $\boldsymbol{\Phi}_{-\tau}(e)$
- ▷ detect intersection with faces of $\mathcal{T}$
- ▷ one-point quadrature on segments



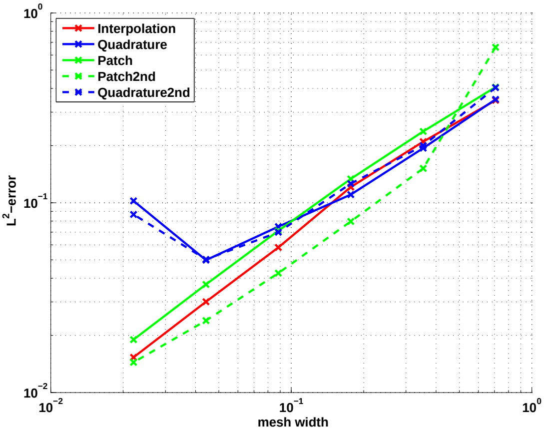
Semi-Lagrangian Interpolation Scheme (II)

Pure transport problem:

$$\frac{\partial \omega}{\partial t} + \mathcal{L}_{\mathbf{v}}\omega = 0 \quad \text{in }]0,1[^2 ,$$

for 1-form $\omega = \omega(\boldsymbol{x}, t)$.

- Triangular edge elements
- Smooth solution
- "Time reversal velocity field"
- $\tau \approx h$



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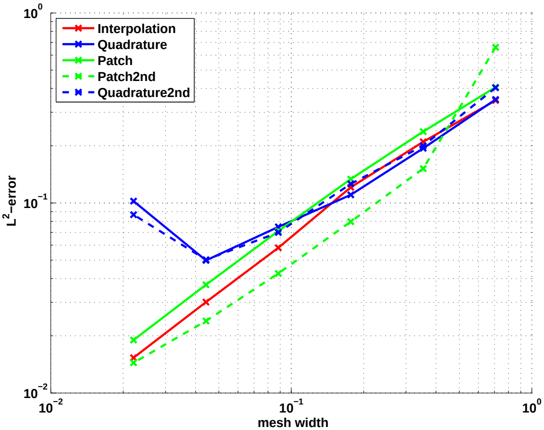

First-order convergence (for smooth ω)
No timestep constraint enforced by stability

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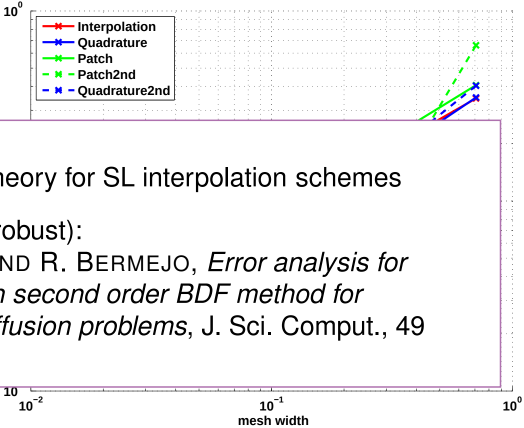

fo

Open problem:

ϵ-robust convergence theory for SL interpolation schemes

For scalar advection (not ϵ-robust):

P. Galán del Sastre and R. Bermejo, *Error analysis for hp-FEM semi-Lagrangian second order BDF method for convection-dominated diffusion problems*, J. Sci. Comput., 49 (2011), pp. 211–237.

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What Next ?

SL for Advection-Diffusion: Convergence (I)

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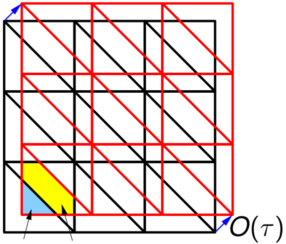

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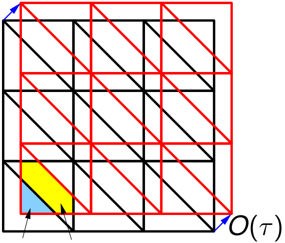

Explicit Euler (with upwinding) is a perturbation of Semi-Lagrange Galerkin

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Main tool: analysis of characteristic methods for stationary advection:

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Theorem: Convergence of *characteristic methods* for stationary advection:

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