Acoustic Localisation of Coronary Artery Disease

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Contents

- Viscoelasticity and wave equations
- Acoustic Localisation of Coronary Artery Disease (CAD)
- High Order (in time) Space-Time FEM

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Viscoelasticity

PDE’s with memory: viscoelasticity

Viscoelasticity

PDE’s with memory
### Main themes

- **Viscoelastic materials exhibit memory**

### Details

**Application areas:**

- damping (polymers)
- structures (concrete)
- porous media (geomechanics)
- electromagnetics (Debye media)
- non-Fickian diffusion
- soft tissue biomechanics
Viscoelasticity

Main themes

- Viscoelastic materials exhibit memory
  - which manifests as:
    - creep

Details

Response:

- $u(t)$
- $F$
- $\tau$
- Fluid
- Solid
Viscoelasticity

Main themes

- Viscoelastic materials exhibit **memory**
- which manifests as:
  - creep
  - relaxation

Details

\[
\begin{align*}
\sigma(t) &\quad \text{stress} \\
\sigma_0 &\quad \text{initial stress} \\
\sigma(t) &\quad \text{response over time} \\
solid &\quad \text{solid material} \\
fluid &\quad \text{fluid material} \\
t &\quad \text{time}
\end{align*}
\]
Viscoelasticity

Main themes

- Viscoelastic materials exhibit **memory**
- which manifests as:
  - creep
  - relaxation
  - hysteresis

Details

![Graph showing stress-strain relationship](image)

**Viscoelasticity**

**Introduction**

**Viscoelasticity**

**Coronary Artery Disease (CAD)**

**Acoustic Localisation of CAD**

**Current Status**

**Next Steps**

**The End**

**BIRS 16w5071: Comp & Num Anal Transient Problems in Acoustics, Elasticity, and Electromagnetism, Jan 17—22, 2016**
Viscoelasticity

Main themes

- Viscoelastic materials exhibit **memory**
- which manifests as:
  - creep
  - relaxation
  - hysteresis
  - frequency dependence

Details

\[
|E(\omega)| = |E'(\omega) + i E''(\omega)|
\]

\[
\tan(\delta) = \frac{E'(\omega)}{E''(\omega)}
\]

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Viscoelasticity

Main themes
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- which manifests as:
  - creep
  - relaxation
  - hysteresis
  - frequency dependence
- Typically described by partial differential equations with either
  - internal variables
  - or memory

Details

\[ u_{tt} - \nabla^2 u = f - \nabla \cdot \sigma \]
\[ \sigma_t + \gamma \sigma = \mu \nabla u \]

\[ u_{tt} - \nabla^2 u = f - \int_0^t b(t-s) \nabla^2 u(s) \, ds \]

Prony: \[ b(t) = \sum_i b_i e^{-t/\tau_i} \]

or weakly singular: \( t^p \)
or fractional calculus

nonlinearity, e.g. \( \gamma \leftarrow \gamma(u) \)
Viscoelasticity

**Main themes**

- Viscoelastic materials exhibit **memory**
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- Typically described by **partial differential equations** with either
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**Details**

\[
\begin{align*}
    u_{tt} - \nabla^2 u &= f - \nabla \cdot \sigma \\
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\end{align*}
\]

\[
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- or weakly singular: \(t^p\)
- or fractional calculus
  - nonlinearity, e.g. \(\gamma \leftarrow \gamma(u)\)

Analysis can exploit the fading memory: an example...

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Maxwell’s equations — dispersive dielectrics

\[ \nabla \times \mathbf{E} + \dot{\mathbf{B}} = 0 \]
\[ \nabla \cdot \mathbf{B} = 0 \]
\[ \nabla \times \mathbf{H} - \dot{\mathbf{D}} = \mathbf{J} \]
\[ \nabla \cdot \mathbf{D} = \varrho \]

What has this got to do with viscoelasticity? Dispersion...
Maxwell’s equations — dispersive dielectrics

\[ \nabla \times E + \dot{B} = 0 \quad \nabla \cdot B = 0 \]
\[ \nabla \times H - \dot{D} = J \quad \nabla \cdot D = \varrho \]

What has this got to do with viscoelasticity? Dispersion...

\[ B = \mu H \] but dielectric polarization is not instantaneous...

Debye: \[ D = \varepsilon_0 (1 + \chi) E + P \] with \[ \tau \dot{P} + P = (\varepsilon_s - \varepsilon_\infty) \varepsilon_0 E \]
with \( P(0) = 0 \) and the \( \varepsilon \)'s known. Hence, we can use either of...

\[ (\varepsilon_0 \varepsilon_\infty)^{-1} D = E + \psi \times E \] or \[ \varepsilon_0 \varepsilon_\infty E = D + \phi \times D \]

which is viscoelasticity! Exploit related results...
Example of a priori control. No Gronwall!

Find $\mathbf{D}$ such that:

$$\ddot{\mathbf{D}} + \nabla \times \mu^{-1} \nabla \times \mathbf{E} + \sigma \dot{\mathbf{E}} = -\dot{\mathbf{J}}_a,$$

where

$$\epsilon \dot{\mathbf{E}}(t) = \dot{\mathbf{D}}(t) + \varphi'(t) \mathbf{D}(0) - \int_0^t \varphi(s)(t - s) \dot{\mathbf{D}}(s) \, ds.$$

**Theorem**

With $\mathbf{G}(t) = -\dot{\mathbf{J}}_a(t) - \sigma \epsilon^{-1} \varphi'(t) \mathbf{D}(0)$ known and if $\sigma > 0$ we have,

$$\mu \epsilon \| \dot{\mathbf{D}}(t) \|_{L_2(0,t;L_2(\Omega))}^2 + \varphi \| \nabla \times \mathbf{D}(t) \|_{L_2(0,t;L_2(\Omega))}^2 + \mu \sigma \varphi \| \dot{\mathbf{D}} \|_{L_2(0,t;L_2(\Omega))}^2 \leq \mu \epsilon \| \dot{\mathbf{D}}(0) \|_{L_2(0,t;L_2(\Omega))}^2 + \| \nabla \times \mathbf{D}(0) \|_{L_2(0,t;L_2(\Omega))}^2 + \frac{\mu \epsilon^2}{\sigma \varphi} \| \mathbf{G} \|_{L_2(0,t;L_2(\Omega))}^2.$$

**Remark:** similar results are possible if $\sigma = 0$.

Using also Rivera & Menzala’s lemma (Quart. Appl. Math., LVII, 1999)
Electromagnetism

Example: a discrete abstract wave equation

Relaxation (fading memory) expressed through internal variable rate equations: find \( u : I \rightarrow V \) such that,

\[
(\rho \dddot{u}(t), v) + a(u(t), v) + b(\dot{u}(t), v) = \langle L(t), v \rangle + \sum_{q=1}^{N_{\varphi}} a(u^*_q(t), v) \quad \forall v \in V,
\]

\[
a(\tau_q \dot{u}^*_q(t) + u^*_q(t), v) = a(\varphi_q u(t), v) \quad \text{for } q = 1, \ldots, N_{\varphi}, \quad \forall v \in V.
\]

And its DG-in-time approximation: find \((U, W) \approx (u, \dot{u})\) such that

\[
\langle \rho \ddot{W}, \vartheta \rangle_n + (\rho [W]_{n-1}, \vartheta^+_n) + a(U, \vartheta)_n + b(W, \vartheta)_n - \sum_{q=1}^{N_{\varphi}} a(Z_q, \vartheta)_n = \langle L, \vartheta \rangle_n
\]

\[
a(\dot{U} - W, \zeta)_n + a([U]_{n-1}, \zeta^+_n) = 0,
\]

for \( q = 1, \ldots, N_{\varphi} \)

\[
a(\tau_q \dot{Z}_q + Z_q - \varphi_q U, \xi_q)_n + a(\tau_q [Z_q]_{n-1}, \xi^+_q, n-1) = 0,
\]

for all test functions \( \theta, \zeta, \xi_1, \ldots \in \mathbb{P}_r(I_n; V^h) \).
Sharp discrete stability

The discrete scheme...

\[
\begin{align*}
\left(\varrho \dot{W}, \vartheta\right)_n + \left(\varrho \left[ W \right]_{n-1}, \vartheta^+_{n-1}\right) + a\left(U, \vartheta\right)_n + b\left(W, \vartheta\right)_n - \sum_{q=1}^{N_\varphi} a\left(Z_q, \vartheta\right)_n &= \left<L, \vartheta\right>_n \\
\left(a\left(\dot{U} - W, \zeta\right)_n + a\left([U]_{n-1}, \zeta^+_{n-1}\right) = 0, \right.
\end{align*}
\]

for \( q = 1, \ldots, N_\varphi \)

\[
a\left(\tau_q \dot{Z}_q + Z_q - \varphi_q U, \xi_q\right)_n + a\left(\tau_q [Z_q]_{n-1}, \xi^+_{q,n-1}\right) = 0,
\]

satisfies, for each time \( t_m \),

\[
\begin{align*}
&\left\| \varrho^{1/2} W_m \right\|_0^2 + \left\| \varphi_0^{1/2} U_m \right\|_V^2 + \sum_{q=1}^{N_\varphi} \left\| \frac{Z_{q,m} - \varphi_q U_m}{\sqrt{\varphi_q}} \right\|_V^2 + 2 \int_0^{t_m} \left( \sum_{q=1}^{N_\varphi} \left\| \frac{Z_q - \varphi_q U}{\sqrt{\tau_q \varphi_q}} \right\|_V^2 + b\left(W, W\right) \right) dt \\
&+ \sum_{n=1}^{m} \left( \sum_{q=1}^{N_\varphi} \left\| \frac{Z_q - \varphi_q U}{\sqrt{\varphi_q}} \right\|_{n-1}^2 + \left\| \varrho^{1/2} \left[ W \right]_{n-1} \right\|_0^2 + \left\| \varphi_0^{1/2} \left[ U \right]_{n-1} \right\|_V^2 \right)
\end{align*}
\]

\[
= 2 \int_0^{t_m} \left< L, W \right> dt + \left\| \varrho^{1/2} W_0^- \right\|_0^2 + \left\| \varphi_0^{1/2} U_0^- \right\|_V^2.
\]

Leads to Non-Gronwall discrete stability. (Error bounds?)
Coronary Artery Disease
Coronary Artery Disease (CAD)

- In the UK in 2010 CAD caused over 14% of all deaths (80,568/561,666).
  - About 95% in people aged 55+ yrs.
- €7.5 billion — approx 2009 cost...
  - €2 billion — healthcare (*per capita* €32)
  - €3.5 billion — lost productivity
  - €2 billion — informal patient care
- An expensive killer — huge tax burden.
- Poor diet & ageing population will exacerbate problem.

What is Coronary Artery Disease?

- Lipids & calcium deposits form atheromatous plaques between endothelium and artery wall
- Stenosis grows and reduces artery calibre.
- Vulnerable plaque suddenly ruptures — causes clot
- Myocardial Ischaemia/Infarction: a ‘heart attack’

What can mathematics offer?

- Biotissue highly viscoelastic and hysteretic
- Acoustic shear waves caused by stenotic wake disturbance travel at frequency-dependent speed
- 150–750 Hz signals detectable at chest surface
- Exploit this for non-invasive computational diagnosis of arterial stenosis via inverse problem
- Challenging and ambitious: at a very early stage...
Localised disturbance: illustrative experiment...

Blood mimicking fluid pumped through artificially stenosed tube constrained within a TMM block (blue, forward; red, reversed).
Coronary Artery Disease (CAD)

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Acoustic Detection of Coronary Artery Disease

John Semmlow¹ and Ketaki Rahalkar²

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²Department of Biomedical Engineering, Rutgers University, Piscataway, New Jersey 08854; email: ketaki_kholkute@yahoo.com

Key Words
phonocardiography, heart sounds, coronary stenosis, coronary bruits, signal processing, cardiac microphones

Abstract
Coronary artery disease (CAD) occurs when the arteries to the heart (the coronary arteries) become blocked by deposition of plaque, depriving the heart of oxygen-bearing blood. This disease is arguably the most important fatal disease in industrialized countries, causing one-third to one-half of all deaths in persons between the ages of 35 and 64 in the United States. Despite the fact that early detection of CAD allows for successful and cost-effective treatment of the disease, the diagnostic accuracy remains low. One of the new approaches to such detection is acoustic localisation of CAD.
Coronary Artery Disease (CAD)


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Coronary Artery Disease (CAD)


one-third to one-half of all deaths in persons between the ages of 50 and 64 in the United States. Despite the fact that early detection of CAD allows for successful and cost-effective treatment of the disease, only 20% of CAD cases are diagnosed prior to a heart attack. The development of a definitive, noninvasive test for detection of coronary blockages is one of the holy grails of diagnostic cardiology. One promising approach to detecting coronary blockages noninvasively is based on identifying acoustic signatures generated by turbulent blood flow through partially occluded coronary arteries. In fact, no other approach to the detection of CAD promises to be as inexpensive, simple to perform, and risk free as the acoustic-based approach. Although sounds associated with partially blocked arteries are easy to identify in more superficial vessels such as the carotids, sounds from coronary arteries are very faint and surrounded by noise such as the very loud valve sounds. To detect these very weak signals requires sophisticated signal processing techniques. This review describes the
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Key Finding...

no other approach ... promises to be as inexpensive, simple... and risk free
What can computational mathematics offer?

Scoping/Feasibility Project: Methods and Aims

- Use the signature chest surface signal for screening and diagnosis. (Expected range $\sim 150 \text{ Hz} - 750 \text{ Hz}$.)
- *In vitro* biomechanics:
  - Characterize tissue mimicking agarose gel.
  - Build gel chest phantoms with controllable ‘stenoses’.
  - Use a fluid loaded model to create shear waves.
- Computational Mathematics:
  - Material characterization through inverse problem data-fitting.
  - Simulations of wave transit through mimicked chest.
  - Stenosis localization through inverse solver.

EVENTUAL AIM

noninvasive screening & diagnosis
Acoustic Localisation of CAD

Speculative Research

Multidisciplinary/international:

- **Blizard (Queen Mary)**: chest phantom construction and experiments
- **CRSC (North Carolina State)**: identification and inverse problem
- **BICOM (Brunel)**: FE models & computation of direct problem

People: C Kruse, JR Whiteman, SE Greenwald, MJ Birch, MP Brewin, J Reeves, HT Banks, ZR Kenz, S Hu, B Kehra, E Cantor, D Mehta, I Gnaneswaren, S Shaw

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Acoustic Localisation of CAD

Schematics - first steps

The reality...
Schematics - first steps

The reality...
Schematics - first steps

The reality... 

Proof of Concept: Agar Gel Chest Phantom

2D idealization

2D is computationally tractable at this early stage
Length scales of interest

Vertical chest cross section.

10cm (approx)

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Figure 4b from Baumüller S, Leschka S, Desbiolles L, et al. Dual-source versus 64-section CT coronary angiography at lower heart rates: comparison of accuracy and radiation dose. Radiology 2009;253:56-64. (c) RSNA 2009
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Vertical chest cross section.

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Acoustic Localisation of CAD

The initial 2D rig – small scale at first

rod and embedded bead
TMM cylinder
programmable ‘shaker’
The mathematical model — classical linear viscodynamics

\[ \rho \ddot{w} - \nabla \cdot \boldsymbol{\sigma} = \rho \mathbf{f} \quad + \text{ initial and boundary data,} \]

with \( \mathbf{w} = \dot{\mathbf{u}} \) and constitutive relationship,

\[ \mathbf{\sigma}(t) = \underbrace{C \varepsilon(w(t))}_{\text{Voigt}} + \underbrace{D \varepsilon(u(t))}_{\text{Hooke}} + \underbrace{D \int_0^t \varphi'(t-s) \varepsilon(u(s)) \, ds}_{\text{Maxwell/Zener}} \]

or internal variables for the Maxwell/Zener term,

\[ z_q(t) := \int_0^t \frac{\varphi_q}{\tau_q} e^{-(t-s)/\tau_q} \mathbf{u}(s) \, ds \]

Linearity!!!!! Why? Justifiable? If not then is there any chance? \ldots
Using the gel cylinder pictured earlier and the three outputs...

Assumptions

For tensors $C$ and $D$ we take as a constitutive law,

$$
\sigma(t) = C\varepsilon(\dot{u}(t)) + D\varepsilon(u(t)) + D \int_0^t \varphi'(t-s)\varepsilon(u(s)) \, ds.
$$

- We assume perfect knowledge of material data by experiments (QMUL/BLT) & inverse problems (CRSC):
  - $E_0 = 229,389$ Pa, $\nu = 0.44$, $E_1 = 55,284$ Pa $\cdot$ s,
  - $G_1 = 3.51$ Pa $\cdot$ s, $\varphi(t) = 1$ and $\rho = 1010$ kg/m$^3$ on the meridian domain: $(0.175, 2.7)$ cm $\times$ $(0, 5.1)$ cm.
- We allow additive/multiplicative Gaussian noise on the signals.
Current Status

Current progress on the localisation problem

Use: gel cylinder, embedded vibrator and three surface outputs...

Algorithm

- A ramped-up sinusoidal 500 Hz axial-displacement source is placed at a known position $\bar{z}$ in the central bore.
- The forward solver computes the axial displacement surface signal at heights 13 mm, 26 mm and 39 mm.
- These ‘truth’ data are ‘banked’ and may or may not be deliberately corrupted with additive noise: $\textit{noisy truth} = \textit{truth} + N_L \times \epsilon$ for $N_L$ a noise level amplitude and $\epsilon \sim N(0, 1)$. (Or with multiplicative noise.)
- The position of the source $\bar{z}$ is now ‘forgotten’.
- MATLAB’s fminsearch iteratively estimates $\bar{z}$ given only the (noisy) truth and the forward solve outputs.

Forward solver: Bicubic Galerkin finite elements were used on a $25 \times 51$ element mesh and with 24,000 Crank-Nicolson time steps for $0 \leq t \leq 1$ s.
Current Status

Some results for a 500 Hz source signal.

Assume perfect knowledge of material constants.
Examples of a ‘true’ signal and a corrupted version: additive Gaussian noise, amplitude $10^{-6}$.
Current Status

Example: 500 Hz with $10^{-6}$ noise

Well posed-ness seems to follow from a good starting value. Some robustness in the presence of significant measurement noise.

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Current Status

Summary: 500 Hz with $10^{-6}$ noise

Well posed-ness seems to follow from a good starting value. Some robustness in the presence of significant measurement noise.
Comparison: 500 Hz no noise (left) $10^{-6}$ noise (right)

Square indicate success: darker shades mean fewer iterations.

Crosses represent failure.

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Comparison: 500 Hz
hi-fi and noiseless (left), lo-fi with $10^{-6}$ noise (right)

Left: $25 \times 51$ mesh of bicubics, 24000 time steps
Right: $25 \times 51$ mesh of bilinears, 12000 time steps
The story so far, and what comes next

- Little change of performance with lo-fi forward solves and very significant additive and multiplicative Gaussian signal-noise.
- Reasonably good initial guesses lead to correct localisation: end effects seem a problem.
- Each localization problem needs about 12 hours: need parallelism, further optimized code, ...  
- Next step is to build and simulate a 3D virtual chest with phantom ribs, lungs, heart, arteries, skin and fat.
High Order DG-in-time FEM

High Order Space-Time FEM for Wave Equations
Space-Time Finite Elements

High order numerical schemes are known for...

- smaller dispersion error in wave propagation problems
- better work/accuracy ratios

We’ve developed **temporally high order time-diagonalised space-time finite element** codes for elasto- and visco-dynamics.

We use continuous **spectral** (i.e. Galerkin with Gauss-Lobatto) FEM in space and **discontinuous Galerkin** in time.

We can compute easily using fifth-degree space-time polynomial approximations.

Hi-Fi solutions are not necessarily important for inverse solvers but speed/parallelism is...
Discretize a wave equation in time with DGFEM: for each \( n = 1, 2, \ldots, N \) in turn, find \((U, W)|_{I_n} \in \mathbb{P}_r(I_n; V) \times \mathbb{P}_r(I_n; V)\) such that

\[
\int_{I_n} (\dot{W}(t), \vartheta(t)) + a(U(t), \vartheta(t)) \, dt + ([W]_{n-1}, \vartheta^+_{n-1}) + a([U]_{n-1}, \zeta^+_{n-1}) \\
+ \int_{I_n} a(\dot{U}(t), \zeta(t)) - a(W(t), \zeta(t)) \, dt = \int_{I_n} \langle L(t), \vartheta(t) \rangle \, dt
\]

\( \forall \vartheta \in \mathbb{P}_r(I_n; V) \) and \( \forall \zeta \in \mathbb{P}_r(I_n; V) \),

with IC’s: \( U_0^- := \ddot{u} \) and \( W_0^- := \ddot{w} \); and \( \mathbb{P}_r(I_n; X) \) the space of polynomials of degree \( r \) on the time interval \( I_n \) with coefficients in the target space \( X \). Note that \( r \) could be \( n \)-dependent.

Diagonalize following: Werder, Gerdes, Schötzau and Schwab, CMAME 2001; 190:6685—6708.
...then gives the decoupled form,

\[ 2\lambda_i(Y_i, \theta) + k_n a(Z_i, \theta) = 2F_i(\theta), \]
\[ a(2\lambda_i Z_i - k_n Y_i, \theta) = 2\beta_i a(U_{n-1}^-, \theta) \]

for \( i = 0, 1, 2, \ldots, r = \) the temporal polynomial degree.

This complex symmetric system requires just one matrix solve for each pair \((Y_i, Z_i)\) and \((U, W)\) are recovered from them.

In IJNME 2014, 98:131156 (DOI: 10.1002/nme.4631), we showed that expected convergence rates are obtained for temporal polynomial degrees up to seven.
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Spectral FE for CG-in-time heat equation

Using Gauss-Lobatto for time integration...

In SISC, 36:B1B13 (DOI: 10.1137/130914589), we showed that expected convergence rates are obtained for temporal polynomial degrees up to six.
Next Steps

Lots to do/test

This is to some extent empirical. Unclear how it deals with

- variable coefficients
- nonlinearities
- dispersion error
- singularities

But it is suited to the coming many-core era...

And one can imagine several variant schemes.
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BIRS 16w5071: Comp & Num Anal Transient Problems in Acoustics, Elasticity, and Electromagnetism, Jan 17—22, 2016
\[ \nabla^2 \phi = 0 \quad \mathcal{H}(F, v)_H = (f, v)_H \quad \forall v \in H \quad z \leftarrow z^2 + c \quad \nabla \times E + \dot{B} = 0 \]

\[ u_t + u \cdot \nabla u = \frac{1}{\rho} \nabla p + \mu \nabla^2 u \quad \frac{\partial C}{\partial t} + \frac{\sigma^2 S^2 \partial C}{2} + \kappa \frac{\partial C}{\partial S} = r C \quad \bar{u} = c^2 \nabla^2 u \]

\[ V_t = e^{-(T-t)} H \Phi(X|F_t) \quad x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \quad -\sigma_{i,j} = f_i \quad (uv)' = uv' + u'v \]

\[ \varepsilon(u) := \frac{1}{2} \left( \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) \quad \Delta u = \lambda u \quad i\kappa \frac{\partial \psi}{\partial t} = -\frac{h}{2\pi} \nabla^2 \psi + V \psi \]

\[ w(u, v) = f(v) \quad \forall v \in H^1(\Omega) \quad \phi(y) = \psi(y) + \int_\Omega \kappa(y-x)\phi(x) \, dx \]

\[ \frac{dy}{dx} = \frac{dy}{du} \frac{du}{dx} \quad \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right) \frac{\partial u}{\partial t} - \nabla^2 u = 0 \]

\[ \nabla \times H - \dot{D} = J \quad \frac{dS}{S} = \mu dt + \sigma dX \quad f(x_0) = \frac{1}{2\pi i} \int \frac{f(z)}{z-x_0} \, dz \]

\[ dG = \left( \frac{\partial}{\partial x} \right) \left( \frac{z}{2\pi i} \int \frac{f(z)}{z-x_0} \, dz \right) \]

A last thought
A last thought

The difference between theory and practice in practice is greater than the difference between theory and practice in theory
Yogi Berra, Albert Einstein, . . .
The difference between theory and practice in practice is greater than the difference between theory and practice in theory. Yogi Berra, Albert Einstein, . . .

Thank You For Listening
Appendix — time permitting
Consider a generic wave equation in weak form: find 
\( u : L_2(0, T) \rightarrow V \) such that

\[
\int_{I_n} (\dot{w}(t), v(t)) + a(u(t), v(t)) \, dt = \int_{I_n} \langle L(t), v(t) \rangle \, dt \quad \forall v \in L_2(0, T; V).
\]

where: \( I_n = (t_{n-1}, t_n) \), \( w = \dot{u} \), \( V \) is a Hilbert space, 
\( a : V \times V \rightarrow \mathbb{R} \) a symmetric bilinear form and \( L : L_2(0, T) \rightarrow V' \) a time dependent linear form containing body loads and boundary tractions.

Discretize in time using Discontinuous Galerkin finite elements...
Discretize in time with DGFEM: for each \( n = 1, 2, \ldots, N \) in turn, find \((U, W)|_{I_n} \in \mathbb{P}_r(I_n; V) \times \mathbb{P}_r(I_n; V)\) such that

\[
\begin{align*}
\int_{I_n} (\dot{W}(t), \vartheta(t)) + a(U(t), \vartheta(t)) \, dt &+ ([W]_{n-1}, \vartheta^+_{n-1}) + a([U]_{n-1}, \zeta^+_{n-1}) \\
+ \int_{I_n} a(\ddot{U}(t), \zeta(t)) - a(W(t), \zeta(t)) \, dt &= \int_{I_n} \langle L(t), \vartheta(t) \rangle \, dt
\end{align*}
\]

\( \forall \vartheta \in \mathbb{P}_r(I_n; V) \) and \( \forall \zeta \in \mathbb{P}_r(I_n; V) \),

with the understanding that the initial conditions are \( U_0^- := \ddot{u} \) and \( W_0^- := \ddot{w} \). Here, for each \( n \), we use \( \mathbb{P}_r(I_n; X) \) to denote the space of polynomials of degree \( r \) on the time interval \( I_n \) with coefficients in the target space \( X \). Note that \( r \) could be \( n \)-dependent.

Diagonalize following: Werder, Gerdes, Schötzau and Schwab, CMAME 2001; 190:6685—6708.
\[
\int_{I_n} \left( \dot{W}(t), \vartheta(t) \right) + a(U(t), \vartheta(t)) \, dt + \left( \langle W \rangle_{n-1}, \vartheta^+_{n-1} \right) + a(\langle U \rangle_{n-1}, \zeta^+_{n-1}) \\
+ \int_{I_n} a(\dot{U}(t), \zeta(t)) \right) - a(W(t), \zeta(t)) \, dt = \int_{I_n} \langle L(t), \vartheta(t) \rangle \, dt \\
\forall \vartheta \in \mathbb{P}_r(I_n; V) \quad \text{and} \quad \forall \zeta \in \mathbb{P}_r(I_n; V),
\]

Let \( \{ \phi_i : i = 0, 1, \ldots, r \} \) be a basis for \( \mathbb{P}_r(I_n) \) and introduce the ansatz forms of the approximations to \( u \) and \( w \) on \( I_n \) as,

\[
U(t)|_{I_n} = \sum_{j=0}^{r} \phi_j(t)U_j \quad \text{and} \quad W(t)|_{I_n} = \sum_{j=0}^{r} \phi_j(t)W_j
\]

where \( \{U_0, U_1, \ldots\}, \{W_0, W_1, \ldots\} \subseteq V \). Replacing each of \( \vartheta(t) \) and \( \zeta(t) \) with \( \phi_i(t)\vartheta \) for \( \phi_i \in \mathbb{P}_r(I_n) \) and \( \vartheta \in V \) we obtain...
\[
\sum_{j=0}^{r} \int_{I_n} \dot{\phi}_j(t) \phi_i(t)(W_j, \vartheta) + \phi_j(t) \phi_i(t) a(U_j, \vartheta) \, dt \\
+ \sum_{j=0}^{r} \phi^+_{j,n-1} \phi^+_{i,n-1}(W_j, \vartheta) = \int_{I_n} \phi_i(t) \langle L(t), \vartheta \rangle \, dt + \phi^+_{i,n-1}(W_{n-1}, \vartheta)
\]

and,

\[
\sum_{j=0}^{r} \int_{I_n} \dot{\phi}_j(t) \phi_i(t) a(U_j, \vartheta) - \phi_j(t) \phi_i(t) a(W_j, \vartheta) \, dt \\
+ \sum_{j=0}^{r} \phi^+_{j,n-1} \phi^+_{i,n-1} a(U_j, \vartheta) = \phi^+_{i,n-1} a(U_{n-1}, \vartheta)
\]

where each holds for all \( \vartheta \in V \) and for each \( i \in \{0, 1, \ldots, r\} \).
Define matrices via,

\[ A_{ij} := \int_{I_n} \dot{\phi}_j(t)\phi_i(t)\,dt + \phi^+_{j,n-1}\phi^+_{i,n-1} \quad \text{and} \quad M_{ij} := \int_{I_n} \phi_j(t)\phi_i(t)\,dt, \]

where \( i \) indexes the rows. Choosing basis functions as the image under the linear map from \([-1, 1]\) to \( I_n \) of the normalized Legendre polynomials gives \( 2M = k_n I \) — diagonal!

A is diagonalizable over \( \mathbb{C} \) for all polynomial degrees of practical interest \((r \leq 100)\). Ref: Werder et al. CMAME 2001; 190:6685—6708.

In fact: \( D = Q^{-1}AQ = \begin{bmatrix} \lambda_0 & \cdots & \lambda_r \end{bmatrix} \) where \( \begin{bmatrix} \cdots \end{bmatrix} \) indicates a diagonal matrix of pairwise complex conjugate eigenvalues and where \( Q \) has complex entries.
Using the summation convention our system is:

\[ A_{ij}(W_j, \vartheta) + \delta_{ij} \frac{k_n}{2} a(U_j, \vartheta) = F_i(\vartheta), \]

\[ A_{ij}a(U_j, \vartheta) - \delta_{ij} \frac{k_n}{2} a(W_j, \vartheta) = G_i(\vartheta), \]

where \( F_i \) and \( G_i \) contain known data.

Let \( \{Y_q\} \) and \( \{Z_q\} \) uniquely solve \( W_j = Q_{jq}Y_q \) and \( U_j = Q_{jq}Z_q \):

\[ A_{ij}Q_{jq}(Y_q, \vartheta) + \delta_{ij} \frac{k_n}{2} Q_{jq}a(Z_q, \vartheta) = F_i(\vartheta), \]

\[ A_{ij}Q_{jq}a(Z_q, \vartheta) - \delta_{ij} \frac{k_n}{2} Q_{jq}a(Y_q, \vartheta) = G_i(\vartheta), \]

Premultiply with \( R = Q^{-1} \), noting that,

\[ R_{pi}A_{ij}Q_{jq} = \delta_{pq} \lambda_p \quad \text{and} \quad R_{pi}\delta_{ij}Q_{jq} = \delta_{pq}, \]

and setting \( F_i(\vartheta) := R_{ip}F_p(\vartheta) \) and \( G_i(\vartheta) := R_{ip}G_p(\vartheta) \) . . .
Time diagonalisation

... then gives the decoupled form,

\[
2\lambda_i(Y_i, \vartheta) + k_n a(Z_i, \vartheta) = 2F_i(\vartheta),
\]
\[
2\lambda_i a(Z_i, \vartheta) - k_n a(Y_i, \vartheta) = 2G_i(\vartheta)
\]

for \( i = 0, 1, 2, \ldots, r = \) the temporal polynomial degree.

This complex symmetric system requires just one matrix solve.

In IJNME 2014, 98:131156 (DOI: 10.1002/nme.4631), we showed that expected convergence rates are obtained for temporal polynomial degrees up to \textbf{seven}.

back
In IJNME 2014; 98:131156 (DOI: 10.1002/nme.4631) we showed that expected convergence rates are obtained for temporal polynomial degrees up to seven.
Using Gauss-Lobatto for time integration...

In SISC, 36:B1B13 (DOI: 10.1137/130914589), we showed that expected convergence rates are obtained for temporal polynomial degrees up to six.