Techniques and estimates to analyze TDBIE

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Idea behind the talk

- Crash course on Laplace and time domain tools for TDBIE
- Galerkin semidiscretization-in-space perfectly integrated in the theoretical framework
- Equations of the first kind
- Summary of tools

A considerable amount of estimates can be obtained as particular cases of a somewhat exotic transmission problem for the wave equation

Achtung. I'm assuming a mathematically motivated audience. This will look like a piece of pure math, but it's just analysis, not algebra or topology. No need to panic.

Reframing the framework

- Everything will be done for the acoustic case
- The elastic case is identical
- Maxwell is not ‘that’ different after all (I used to think it was)
- Actual transmission problems can be dealt with using a two-field formulation
- Coupling with FEM-like discretization is also doable (it’s done)
- Wave-structure interaction is doable (also done)
- To be done:
  - regularity theory and approximation theorems
  - equations of the second kind (?)
  - free coupling of FEM and BEM

The self-centered mathematician apology playbook

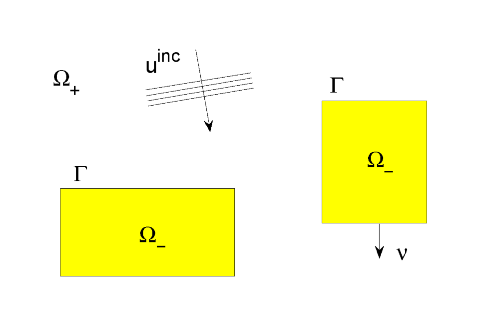
I'm not citing (almost) anyone. However, this work condenses CQ- and Galerkin-TDBIE flavored results from the literature:

- Bamberger-HaDuong and related work (Becache, Bachelot, Terrasse, etc)
- German speaking CQ revival (Sauter, Hackbusch, Banjai, Schanz, López Fdez, Melenk)
- Delaware bunch (Monk, Weile, myself and my students)

A modest goal. Can we find a way to make many techniques real simple so that we can move to more complicated (theoretical) issues without being repetitive? At the end this looks like an exercise. (Of course, once you know the result...) But that's kind of our goal.

A conversation (in the 90s). My adviser (M.C.): make the proof so simple so that it cannot be published. Me (mentally): I kind of want a job.

Geometry and notation



- Traces: $\gamma^{\pm} u$
- Normal traces: $\gamma_\nu^{\pm} \mathbf{v}$
- Normal derivatives: $\partial_\nu^{\pm} := \gamma_\nu^{\pm} \nabla$
- Jumps: $[\![\gamma u]\!] := \gamma^- u - \gamma^+ u$
- etc

Geometrically (analytically) speaking, we only need a fully functioning trace operator (and a normal vector field on the boundary)

Ansatz and observation spaces

Trial-test spaces

$$X_h \subset H^{-1/2}(\Gamma) = H^{1/2}(\Gamma)^* \qquad Y_h \subset H^{1/2}(\Gamma) = H^{-1/2}(\Gamma)^*$$

Polar sets = Galerkin observation

For instance

$$\beta \in X_h^\circ \qquad \text{means} \qquad \langle \beta, \mu^h \rangle_\Gamma = 0 \quad \forall \mu^h \in X_h.$$

But... admit the chance that $X_h = H^{-1/2}(\Gamma)$, or $X_h = \{0\}$, or $X_h = \widetilde{H}^{-1/2}(\Gamma_{\rm scr})$.

A set of compatible transmission conditions

You are allowed four data

$$[\![\gamma u]\!] - \alpha_1 \in Y_h \qquad \gamma^+ u - \alpha_2 \in X_h^\circ$$

$$[\![\partial_\nu u]\!] - \beta_1 \in X_h \qquad \partial_\nu^- u - \beta_2 \in Y_h^\circ$$

$$\begin{aligned}(\nabla u, \nabla v)_{\mathbb{R}^d \setminus \Gamma} + (\Delta u, v)_{\mathbb{R}^d \setminus \Gamma} &= \langle \partial_\nu^- u, \gamma^- v \rangle_\Gamma - \langle \partial_\nu^+ u, \gamma^+ v \rangle_\Gamma \\ &= \langle \partial_\nu^- u, [\![\gamma v]\!] \rangle_\Gamma + \langle [\![\partial_\nu u]\!], \gamma^+ v \rangle_\Gamma\end{aligned}$$

Remark. You can obviously use γ^- and ∂_ν^+ instead.

Some insight: conservation of energy

If for all t (the Laplacian is applied in $\mathbb{R}^d \setminus \Gamma$)

$$\begin{aligned} \ddot{u}(t) &= \Delta u(t), & \\ [\![\gamma u]\!](t) &\in Y_h & \gamma^+ u(t) &\in X_h^\circ \\ [\![\partial_\nu u]\!](t) &\in X_h & \partial_\nu^- u(t) &\in Y_h^\circ \end{aligned}$$

then

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t} & \left(\tfrac{1}{2} \|\nabla u(t)\|^2_{\mathbb{R}^d\setminus\Gamma} + \tfrac{1}{2} \|\dot{u}(t)\|^2_{\mathbb{R}^d\setminus\Gamma} \right) \\ &= (\nabla u(t), \nabla \dot{u}(t))_{\mathbb{R}^d\setminus\Gamma} + (\ddot{u}(t), \dot{u}(t))_{\mathbb{R}^d\setminus\Gamma} \\ &= (\nabla u(t), \nabla \dot{u}(t))_{\mathbb{R}^d\setminus\Gamma} + (\Delta u(t), \dot{u}(t))_{\mathbb{R}^d\setminus\Gamma} \\ &= \langle \partial_\nu^- u(t), [\![\gamma \dot{u}(t)]\!] \rangle_\Gamma + \langle [\![\partial_\nu u(t)]\!], \gamma^+ \dot{u}(t) \rangle_\Gamma \\ &= 0 \end{aligned}$$

A student after class (in 2012). Was that ... like ... a proof?

A well-posed elliptic problem

PDE form

$$U = \Delta U$$

$$[\![\gamma U]\!] - \alpha_1 \in Y_h \qquad \gamma^+ U - \alpha_2 \in X_h^\circ$$

$$[\![\partial_\nu U]\!] - \beta_1 \in X_h \qquad \partial_\nu^- U - \beta_2 \in Y_h^\circ$$

Its variational formulation

$$U \in H^1(\mathbb{R}^d \setminus \Gamma)$$

$$[\![\gamma U]\!] - \alpha_1 \in Y_h \qquad \gamma^+ U - \alpha_2 \in X_h^\circ$$

$$(\nabla U, \nabla V)_{\mathbb{R}^d \setminus \Gamma} + (U, V)_{\mathbb{R}^d} = \langle \beta_2, [\![\gamma V]\!] \rangle_\Gamma + \langle \beta_1, \gamma^+ V \rangle_\Gamma$$

$$\forall V \in H^1(\mathbb{R}^d \setminus \Gamma), \qquad [\![\gamma V]\!] \in Y_h \qquad \gamma^+ V \in X_h^\circ$$

Remark. This problem is perfectly coercive, right? There's more. You can bound its solution **independently of** h.

A wave equation in the Laplace domain

PDE form

$$s^2 U = \Delta U$$

$$[\![\gamma U]\!] - \alpha_1 \in Y_h \qquad \gamma^+ U - \alpha_2 \in X_h^\circ$$

$$[\![\partial_\nu U]\!] - \beta_1 \in X_h \qquad \partial_\nu^- U - \beta_2 \in Y_h^\circ$$

VF

$$U \in H^1(\mathbb{R}^d \setminus \Gamma)$$

$$[\![\gamma U]\!] - \alpha_1 \in Y_h \qquad \gamma^+ U - \alpha_2 \in X_h^\circ$$

$$(\nabla U, \nabla V)_{\mathbb{R}^d \setminus \Gamma} + s^2 (U, V)_{\mathbb{R}^d} = \langle \beta_2, [\![\gamma V]\!] \rangle_\Gamma + \langle \beta_1, \gamma^+ V \rangle_\Gamma$$

$$\forall V \in H^1(\mathbb{R}^d \setminus \Gamma), \qquad [\![\gamma V]\!] \in Y_h \qquad \gamma^+ V \in X_h^\circ$$

Remark. Boundary data are now applied as instantaneous impulses at time $t = 0$.

A wave equation ... the key idea

$$U \in H^1(\mathbb{R}^d \setminus \Gamma)$$
$$[\![\gamma U]\!] - \alpha_1 \in Y_h \qquad \gamma^+ U - \alpha_2 \in X_h^\circ$$
$$(\nabla U, \nabla V)_{\mathbb{R}^d \setminus \Gamma} + s^2 (U, V)_{\mathbb{R}^d} = \langle \beta_2, [\![\gamma V]\!] \rangle_\Gamma + \langle \beta_1, \gamma^+ V \rangle_\Gamma \quad \forall V \ldots$$

The essential TC are taken care by the **optimal (non-physical) lifting** of Bamberger & HaDuong

$$U_d \in H^1(\mathbb{R}^d \setminus \Gamma)$$
$$[\![\gamma U_d]\!] = \alpha_1 \qquad \gamma^+ U_d = \alpha_2$$
$$(\nabla U_d, \nabla V)_{\mathbb{R}^d \setminus \Gamma} + |s|^2 (U_d, V)_{\mathbb{R}^d} = 0 \qquad \forall V \in H_0^1(\mathbb{R}^d \setminus \Gamma)$$

Remark. You then study the problem satisfied by $U_0 = U - U_d$ (which can be used as test function!)

Laplace's puzzle: the corner pieces

If

$$a(U,V;s) := (\nabla U, \nabla V)_{\mathbb{R}^d\setminus\Gamma} + s^2 (U,V)_{\mathbb{R}^d}$$

and

$$|\!|\!| U |\!|\!|_{|s|}^2 := a(U,\overline{U};|s|) = \|\nabla U\|_{\mathbb{R}^d\setminus\Gamma}^2 + \|s\,U\|_{\mathbb{R}^d}^2$$

is the energy norm, then

$$|a(U,V;s)| \le |\!|\!| U |\!|\!|_{|s|} |\!|\!| V |\!|\!|_{|s|} \qquad \text{(boundedness)}$$

$$\mathrm{Re}\, a(U,\overline{s\,U};s)| = (\mathrm{Re}\, s) |\!|\!| U |\!|\!|_{|s|}^2 \qquad \text{(coercivity)}$$

$$\min\{1,\mathrm{Re}\, s\} \|U\|_{1,\mathbb{R}^d\setminus\Gamma} \le |\!|\!| U |\!|\!|_{|s|} \le \frac{|s|}{\min\{1,\mathrm{Re}\, s\}} \|U\|_{1,\mathbb{R}^d\setminus\Gamma}$$

Remark. The coercivity identity is the one telling you to differentiate one of the arguments in the weak TD formulation

Laplace's puzzle: the impossible blue sky part

If

$$-\Delta U + |s|^2 U = 0 \quad \text{in } \mathbb{R}^d \setminus \Gamma, \qquad \gamma^\pm U = \alpha_\pm,$$

then

$$\|\!| U \|\!|_{|s|} \le C_\Gamma \max\{1, |s|\}^{1/2} \|\alpha_\pm\|_{1/2,\Gamma}.$$

Therefore, if

$$-\Delta U + s^2 U = 0 \quad \text{in } \mathbb{R}^d \setminus \Gamma,$$

then

$$\|\partial_\nu^\pm U\|_{-1/2,\Gamma} \le C_\Gamma \left(\frac{|s|}{\min\{1, \operatorname{Re} s\}} \right)^{1/2} \|\!| U \|\!|_{|s|}$$

Remark. Maxwell differs here

The finished image can now be guessed

- You can get Laplace domain bounds for the associated transfer function. With these, use any Inversion of the L.T. theorem to get time domain results.
- You can focus on coercivity results and apply Plancherel. This gives a theoretical basis for some weighted Galerkin methods

Remark. We leave Laplace finishing the puzzle and move a more dynamical view

A dynamical system

Second order form. For all t

$$\begin{aligned} \ddot{u}(t) &= \Delta u(t) \\ [\![\gamma u]\!](t) - \alpha_1(t) &\in Y_h \qquad & \gamma^+ u(t) - \alpha_2(t) &\in X_h^\circ \\ [\![\partial_\nu u]\!](t) - \beta_1(t) &\in X_h \qquad & \partial_\nu^- u(t) - \beta_2(t) &\in Y_h^\circ \end{aligned}$$

First order form. For all t

$$\begin{aligned} \dot{u}(t) &= \nabla \cdot \mathbf{v}(t) \qquad & \dot{\mathbf{v}}(t) &= \nabla u(t) \\ [\![\gamma u]\!](t) - \alpha_1(t) &\in Y_h \qquad & \gamma^+ u(t) - \alpha_2(t) &\in X_h^\circ \\ [\![\gamma_\nu \mathbf{v}]\!](t) - \partial^{-1}\beta_1(t) &\in X_h \qquad & \gamma_\nu^- \mathbf{v}(t) - \partial^{-1}\beta_2(t) &\in Y_h^\circ \end{aligned}$$

Main advantage. All TC are now essential. The lifting will take care of all of them at the same time. Note how natural data have been integrated in time

The main decomposition

Write

$$(u(t), \mathbf{v}(t)) = (u_d(t), \mathbf{v}_d(t)) + (u_0(t), \mathbf{v}_0(t)),$$

where (elliptic lifting of TCs)

$$\begin{aligned} u_d(t) &= \nabla \cdot \mathbf{v}_d(t) & \mathbf{v}_d(t) &= \nabla u_d(t) \\ [\![\gamma u_d]\!](t) - \alpha_1(t) &\in Y_h & \gamma^+ u_d(t) - \alpha_2(t) &\in X_h^\circ \\ [\![\gamma_\nu \mathbf{v}_d]\!](t) - \partial^{-1}\beta_1(t) &\in X_h & \gamma_\nu^- \mathbf{v}_d(t) - \partial^{-1}\beta_2(t) &\in Y_h^\circ \end{aligned}$$

and (non-homogeneous ODE in Hilbert space)

$$\begin{aligned} \dot{u}_0(t) &= \nabla \cdot \mathbf{v}_0(t) - \dot{u}_d(t) + u_d(t) & \dot{\mathbf{v}}_0(t) &= \nabla u_0(t) - \dot{\mathbf{v}}_d(t) + \mathbf{v}_d(t) \\ [\![\gamma u_0]\!](t) &\in Y_h & \gamma^+ u_0(t) &\in X_h^\circ \\ [\![\gamma_\nu \mathbf{v}_0]\!](t) &\in X_h & \gamma_\nu^- \mathbf{v}_0(t) &\in Y_h^\circ \end{aligned}$$

And... honestly, that's all it takes to study this problem. (Well, up to a point.)

A problem to rule them all

How's this related to integral equations?

Let me copy-paste this problem again: for all $t \geq 0$

$$
\begin{aligned}
\ddot{u}(t) &= \Delta u(t) \\
[\![\gamma u]\!](t) - \alpha_1(t) &\in Y_h \qquad \gamma^+ u(t) - \alpha_2(t) \in X_h^\circ \\
[\![\partial_\nu u]\!](t) - \beta_1(t) &\in X_h \qquad \partial_\nu^- u(t) - \beta_2(t) \in Y_h^\circ
\end{aligned}
$$

We now pick X_h, Y_h, and different choices of data. Quantities of interest:

$$
\|u(t)\|_{1,\mathbb{R}^d}, \qquad \|\gamma^\pm u(t)\|_{1/2,\Gamma}, \qquad \|\partial_\nu^\pm u(t)\|_{-1/2,\Gamma} = \|\gamma_\nu^\pm \dot{\mathbf{v}}(t)\|_{-1/2,\Gamma}
$$

Luckily... when you obtain bounds for $u(t)$ and $\mathbf{v}(t)$, you also produce bounds for their first derivatives

Examples galore

Densities given, no observation

$X_h = \{0\}$, $Y_h = \{0\}$ (polar sets do not give any information!)

$$\begin{aligned} \ddot{u}(t) &= \Delta u(t) \\ [\![\gamma u]\!](t) &= \varphi(t) \\ [\![\partial_\nu u]\!](t) &= \lambda(t) \end{aligned}$$

so

$$u = \mathcal{S} * \lambda - \mathcal{D} * \varphi$$

and we also gets bounds for all the operators of the acoustic Calderón projector.

Find the density by looking at the trace

$X_h = H^{-1/2}(\Gamma)$ (entire space to look into and $X_h^\circ = \{0\}$),
$Y_h = \{0\}$ (the polar set is $H^{-1/2}(\Gamma)$)

$$\begin{aligned} \ddot{u}(t) &= \Delta u(t) \\ [\![\gamma u]\!](t) &= 0 \\ \gamma^+ u(t) &= \alpha(t) \\ \lambda(t) := [\![\partial_\nu u]\!](t) &\in H^{-1/2}(\Gamma) \qquad \text{(unknown)} \end{aligned}$$

so we are solving

$$\mathcal{V} * \lambda = \alpha, \qquad u = \mathcal{S} * \lambda$$

(Indirect method for Dirichlet problem. Single layer potential representation.)

Same spaces, different data

$X_h = H^{-1/2}(\Gamma)$ (entire space to look into and $X_h^\circ = \{0\}$),
$Y_h = \{0\}$ (the polar set is $H^{-1/2}(\Gamma)$)

$$\begin{aligned} \ddot{u}(t) &= \Delta u(t) \\ [\![\gamma u]\!](t) &= \varphi(t) \\ \gamma^+ u(t) &= 0 \\ \lambda(t) := [\![\partial_\nu u]\!](t) &\in H^{-1/2}(\Gamma) \qquad \text{(unknown)} \end{aligned}$$

so we are solving

$$\mathcal{V} * \lambda = \tfrac{1}{2}\varphi + \mathcal{K} * \varphi, \qquad u = \mathcal{S} * \lambda - \mathcal{D} * \varphi$$

(Direct formulation for interior Dirichlet problem. Note that $\gamma^- u = \varphi$. Small variant for exterior problem.)

Find the density by looking at the trace (discretely)

X_h (finite dimensional), $Y_h = \{0\}$

$$\begin{aligned} \ddot{u}(t) &= \Delta u(t) \\ [\![\gamma u]\!](t) &= 0 \\ \gamma^+ u(t) - \alpha(t) &\in X_h^\circ \\ \lambda^h(t) := [\![\partial_\nu u]\!](t) &\in X_h \qquad \text{(unknown)} \end{aligned}$$

so we are solving

$$\lambda^h \in X_h \quad \langle \mathcal{V} * \lambda^h - \alpha, \mu \rangle_\Gamma = 0 \quad \forall \mu \in X_h$$

and then plugging

$$u = \mathcal{S} * \lambda^h$$

(The discretization of the direct formulation is similar.)

Same spaces, different data

X_h (finite dimensional), $Y_h = \{0\}$

$$\begin{aligned} \ddot{u}(t) &= \Delta u(t) \\ [\![\gamma u]\!](t) &= 0 \\ \gamma^+ u(t) &\in X_h^\circ \\ -\lambda^h(t) := [\![\partial_\nu u]\!](t) - \lambda(t) &\in X_h \qquad \text{(unknown)} \end{aligned}$$

so we are solving

$$\lambda^h \in X_h \quad \langle \mathcal{V} * \lambda^h - \mathcal{V} * \lambda, \mu \rangle_\Gamma = 0 \quad \forall \mu \in X_h$$

and then plugging

$$u = \mathcal{S} * (\lambda - \lambda^h)$$

I could go on and on and on

- Deactivating X_h ($X_h = \{0\}$) we obtain double-layer potential related integral equations (Neumann problem) and their Galerkin semidiscretizations
- Activating X_h and Y_h at different parts of the boundary we obtain the symmetric Galerkin method for mixed boundary value problems (scatterers with different material properties)
- Activating X_h in one part of Γ and deactivating Y_h, we get Dirichlet screens and their discretizations. (And Neumann screens? Guess!)
- Integral equations of the second kind also follow from this theory. However (!!), there doesn't to be any conclusion about their semidiscretization in the pool.

Shameless product placement

This book will look really nice on your bookshelf