Random Partitions and the Quantum Benjamin-Ono Hierarchy

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BIRS Workshop on β-Ensembles:
Universality, Integrability, and Asymptotics

13 April 2016

Outline

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- **I.** Random Partitions from Quantum Benjamin-Ono
- **II.** Limit Shapes and Gaussian Fluctuations
- **III.** Integrability of Quantum Benjamin-Ono
- **IV.** Poisson at High Frequency
- **V.** A β-Refined Topological Recursion

$\star$ QBO $\Longleftrightarrow$ Circular β-Ensembles at $N \to \infty$ $\star$

Circular β-Ensembles: $\beta = 2$

Given $V : \mathbb{T} \to \mathbb{R}$ potential on $\mathbb{T} = \{w \in \mathbb{C} : |w| = 1\}$,

Random Matrices: class functions on $U(N)$ lead to

$$Z_{\mathbb{T};V}(N|2,t) \;=\; \oint_{\mathbb{T}^N} e^{-\frac{N}{t}\sum_{i=1}^{N} V(w_i)} \prod_{i<j} \left|w_i - w_j\right|^2 \prod_{i=1}^{N} \frac{dw_i}{2\pi \mathbf{i} w_i}$$

Gessel-Heine-Szegö-Weyl: Z is a *Toeplitz determinant*

$$\det T_N\left(e^{-\frac{N}{t}V(w)}\right) = Z_{\mathbb{T};V}(N|2,t) = \Pi \cdot \mathbb{P}\left(\lambda_1' \le N\right).$$

and *law of first column* of λ from <u>Schur measure</u> $M_V(-\varepsilon, \varepsilon)$.

- OPUC from $e^{-\frac{N}{t}V(w)}$ vs. Schurs $s_\lambda(\vec{w})$ from $\prod_{i<j}|w_i - w_j|^2$
- Random Matrices $\to$ Random Partitions: N *formal variable!*

Circular β-Ensembles: **Integrability**

Quantum Calogero-Sutherland Hamiltonian

$$\widehat{\mathcal{Y}}_{3;N} := -\frac{1}{2}\sum_{i=1}^{N}\frac{\partial^2}{\partial x_i^2} + \frac{\beta}{2}\Big(\frac{\beta}{2}-1\Big)\sum_{i<j}\frac{1}{\big(\frac{L}{\pi}\sin\frac{\pi}{L}(x_i-x_j)\big)^2}$$

For L-periodic boundary conditions, take $w = e^{2\pi \mathbf{i} x/L}$ on

$$\mathbb{T} := \{w \in \mathbb{C} \ : \ |w| = 1\}$$

$\widehat{\mathcal{Y}}_{3;N}$ has a multi-valued ground state

$$\Psi_\circ(w_1,\ldots,w_N|\beta) := \prod_{i<j}(w_i - w_j)^{\beta/2}$$

with amplitude $\big|\Psi_\circ(\vec{w}|\beta)\big|^2 = \prod_{i<j}\big|w_i - w_j\big|^\beta$.

- Forrester *Random Matrices, Log-Gases, and the Calogero-Sutherland Model* (1998)

Circular β-Ensembles: **Integrability**

Given $V : \mathbb{T} \to \mathbb{R}$ background potential

$$V(w) = \sum_{k=1}^{\infty} \frac{V_k w^k}{k} + \frac{\overline{V}_k w^{-k}}{k}$$

define partition function

$$Z_{\mathbb{T};V}(N|\beta, t) = \oint_{\mathbb{T}^N} e^{-\frac{N}{t}\sum_{i=1}^{N} V(w_i)} \prod_{i<j} \left|w_i - w_j\right|^{\beta} \prod_{i=1}^{N} \frac{dw_i}{2\pi \mathbf{i} w_i}$$

1. **Know:** law depends only on $\rho = \frac{1}{N}\sum_{i=1}^{N} \delta(w - w_i)$
2. **Know:** for $V \equiv 0$, law invariant under *hierarchy* of commuting flows $\widehat{\mathcal{Y}}_{\ell;N}$ diagonalized on "Jacks"$\cdot\Psi_\circ$
3. **New:** can write *hierarchy* $\Psi_\circ^{-1}\widehat{\mathcal{Y}}_{\ell;N}\Psi_\circ$ only through ρ
4. ***Result:*** dense Gaussian matrix model for Jack measures of random partitions $\Leftrightarrow$ circular β-ensembles arbitrary V

Outline

I. Random Partitions from Quantum Benjamin-Ono

Take $v(x) \equiv v(x+2\pi)$ in *classical Benjamin-Ono equation*

$$v_t + v v_x = -\varepsilon_1 \mathcal{J}[v_{xx}]$$

As $\varepsilon_1 \to 0$ $\quad v_t + v v_x = 0$ the *classical Hopf equation*.

For $w = e^{\mathbf{i}x}$, at $t = 0$

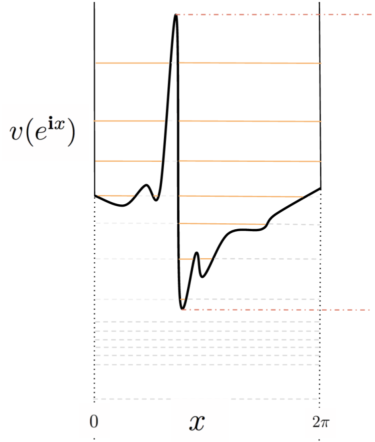
$$v(w) = \sum_{k=1}^{\infty} \left(V_k w^{-k} + \overline{V_k} w^k \right)$$

$$\mathbb{T} := \{ w \in \mathbb{C} \;:\; |w| = 1 \}$$

Periodic Hilbert transform $\mathcal{J}$ polarizes $v : \mathbb{T} \to \mathbb{R}$

$$v(w) = v_-(w) + v_+(w)$$

by $\mathcal{J} v_\pm = \pm \mathbf{i} v_\pm$.



I. Random Partitions from Quantum Benjamin-Ono

Throughout

$$\varepsilon_2 < 0 < \varepsilon_1 \,.$$

Define $\langle \cdot, \cdot \rangle_{-\varepsilon_1 \varepsilon_2}$ on

$$\mathcal{F} := \bigotimes_{k=1}^{\infty} \mathbb{C}[\mathcal{V}_k]$$

by declaring

$$\mathcal{V}_{-k} := -\varepsilon_1 \varepsilon_2 \, k \frac{\partial}{\partial \mathcal{V}_k}$$

to adjoint of $M(\mathcal{V}_k)$.

- Canonically quantize $\{V_{-k}, V_{k'}\} = \mathbf{i} k \delta(k - k')$ in $\{V_0 = 0\}$

I. Random Partitions from Quantum Benjamin-Ono

Using $\mathcal{V}_{-k} := -\varepsilon_1\varepsilon_2 \, k \frac{\partial}{\partial \mathcal{V}_k}$, **degree operator**

$$\mathcal{Y}_2 := \sum_{k=1}^{\infty} \mathcal{V}_k \mathcal{V}_{-k}$$

decomposes

$$\bigotimes_{k=1}^{\infty} \mathbb{C}[\mathcal{V}_k] = \mathcal{F} = \bigoplus_{d=0}^{\infty} \mathcal{F}[d]$$

into $\dim \mathcal{F}[d] < \infty$. Orthogonal basis

$$\mathcal{V}_\mu := \mathcal{V}_1^{\#_1} \mathcal{V}_2^{\#_2} \cdots \mathcal{V}_k^{\#_k} \cdots$$

Partitions $\mu := (\#_1, \#_2, \ldots) \in \mathbb{N}^\infty$ with

$$\deg \mu := \sum_{k=1}^{\infty} k \#_k < \infty.$$

I. Random Partitions from Quantum Benjamin-Ono

The *Quantum Benjamin-Ono Hamiltonian*

$$\mathcal{Y}_3(\varepsilon_2, \varepsilon_1) := \sum_{h_1, h_2 = 0}^{\infty} \mathcal{V}_{h_1} \mathcal{V}_{h_2 - h_1} \mathcal{V}_{-h_2} + (\varepsilon_1 + \varepsilon_2) \sum_{h=0}^{\infty} h \mathcal{V}_h \mathcal{V}_{-h}$$

defined by $\mathcal{V}_{-k} := -\varepsilon_1 \varepsilon_2 \, k \frac{\partial}{\partial \mathcal{V}_k}$ mixes up

$$\mathcal{F} = \bigotimes_{k=1}^{\infty} \mathbb{C}[\mathcal{V}_k].$$

As $\varepsilon_1 + \varepsilon_2 \to 0$, get *Quantum Hopf Hamiltonian* $\mathcal{Y}_3(-\varepsilon, \varepsilon)$.

1. $\mathcal{Y}_3$ is self-adjoint for $\langle \cdot, \cdot \rangle_{-\varepsilon_1 \varepsilon_2}$
2. $\mathcal{Y}_3$ commutes with the degree operator $\mathcal{Y}_2$
3. $\mathcal{Y}_2$ has finite-dimensional eigenspaces on $\mathcal{F}$

I. Random Partitions from Quantum Benjamin-Ono

Spectral Theorem: $\mathcal{F}$ has a homogeneous orthogonal basis

$$P_\lambda(\mathcal{V}|\,\varepsilon_2, \varepsilon_1)$$

of eigenfunctions of quantum B-O Hamiltonian $\mathcal{Y}_3(\varepsilon_2, \varepsilon_1)$.
These are the *Jack special functions*.

- $\mathcal{V}_\mu \in \mathcal{F}$ indexed by *partitions* $\mu = 1^{\#_1} 2^{\#_2} \cdots k^{\#_k}$
- $P_\lambda \in \mathcal{F}$ indexed by *partitions* $\lambda \in \mathbb{Y}_d$ of degree d

$$0 \leq \cdots \leq \lambda_2 \leq \lambda_1 \qquad \deg(\lambda) := \sum_{i=1}^{\infty} \lambda_i < \infty$$

- inner product $-\varepsilon_1\varepsilon_2$ vs. anisotropy $\varepsilon_1 + \varepsilon_2$
- compare $\frac{\beta}{2} = -\frac{\varepsilon_2}{\varepsilon_1} = \frac{1}{\alpha}$ inverse Jack parameter

As $\varepsilon_1 + \varepsilon_2 \to 0$, Jacks become *Schur functions* $P_\lambda(\mathcal{V}|-\varepsilon, \varepsilon)$

I. Random Partitions from Quantum Benjamin-Ono

Generalities: Have $\mathcal{F}$ with $\langle \cdot, \cdot \rangle_{\varepsilon}$ and orthogonal basis $P_\lambda(\mathcal{V}|\varepsilon)$ of eigenfunctions of $\mathcal{Y}(\varepsilon)$. Completion of $\mathcal{F}$ has

$$\Psi(\mathcal{V}) = \sum_{\lambda} \widehat{\Psi}(\lambda|\varepsilon) \frac{P_\lambda(\mathcal{V}|\varepsilon)}{||P_\lambda(\mathcal{V}|\varepsilon)||_{\varepsilon}}$$

states Ψ satisfying

$$\langle \Psi | \Psi \rangle_{\varepsilon} = \sum_{\lambda} \left| \widehat{\Psi}(\lambda|\varepsilon) \right|^2 < \infty.$$

The **random energy distribution of** Ψ **w.r.t.** $\mathcal{Y}(\varepsilon)$ is

$$\mathrm{Prob}_{\Psi;\mathcal{Y}(\varepsilon)}(\lambda) := \frac{1}{||\Psi||_{\varepsilon}^2} \left| \widehat{\Psi}(\lambda|\varepsilon) \right|^2$$

a <u>random index</u> λ of eigenfunctions of the Hamiltonian $\mathcal{Y}(\varepsilon)$.

I. Random Partitions from Quantum Benjamin-Ono

Data: Have $\mathcal{F}$, $\langle \cdot, \cdot \rangle_{-\varepsilon_1\varepsilon_2}$, orthogonal basis $P_\lambda(\mathcal{V}|\,\varepsilon_2, \varepsilon_1)$ of eigenfunctions of quantum B-O Hamiltonian $\mathcal{Y}_3(\varepsilon_2, \varepsilon_1)$.

For $V_k \in \mathbb{C}$, take ***coherent state*** at each $|0\rangle_k = 1 \in \mathbb{C}[\mathcal{V}_k]$

$$\Psi_{v,-\varepsilon_1\varepsilon_2} := \prod_{k=1}^{\infty} \exp\left(\frac{V_k \mathcal{V}_k}{\sqrt{-\varepsilon_1\varepsilon_2}} \right) |0\rangle_k .$$

The random energy distribution of $\Psi_{v,-\varepsilon_1\varepsilon_2}$ w.r.t. $\mathcal{Y}_3(\varepsilon_2, \varepsilon_1)$

$$\mathrm{Prob}_{\Psi;\mathcal{Y}(\varepsilon)}(\lambda) := \frac{1}{\langle \Psi_v, \Psi_v \rangle_{-\varepsilon_1\varepsilon_2}} \left| \frac{P_\lambda(V|\,\varepsilon_2, \varepsilon_1)}{||P_\lambda(\cdot|\,\varepsilon_2, \varepsilon_1)||_{-\varepsilon_1\varepsilon_2}} \right|^2$$

defines **Jack measure** $M_v(\varepsilon_2, \varepsilon_1)$ for every $\varepsilon_2 < 0 < \varepsilon_1$ and

$$v(w) = \sum_{k=1}^{\infty} \left(V_k w^k + \overline{V_k} w^{-k} \right).$$

I. Random Partitions from Quantum Benjamin-Ono

Stanley's Cauchy identity: Jack measures $M_v(\varepsilon_2, \varepsilon_1)$ on λ

$$\prod_{k=1}^{\infty} \exp\left(\frac{\overline{V_k} V_k}{-\varepsilon_1 \varepsilon_2 k} \right) = \Pi(\overline{V}, V | \tfrac{1}{-\varepsilon_1 \varepsilon_2}) = \sum_{\lambda \in \mathbb{Y}} \frac{|P_\lambda(V|\varepsilon_2, \varepsilon_1)|^2}{\langle P_\lambda, P_\lambda \rangle_{-\varepsilon_1 \varepsilon_2}}$$

unify β, V deformations of *Poissonized Plancherel measures:*

$$\begin{tikzcd}
 & M_v(\varepsilon_2, \varepsilon_1) \arrow[dl] \arrow[dr] & \\
M_\bullet(\varepsilon_2, \varepsilon_1) \arrow[dr] & & M_v(-\varepsilon, \varepsilon) \arrow[dl] \\
 & M_\bullet(-\varepsilon, \varepsilon) &
\end{tikzcd}$$

$(\varepsilon_2, \varepsilon_1) \longrightarrow (-\varepsilon, \varepsilon)$ Schur measures Okounkov (1999)

$v_\bullet(w) = w + \frac{1}{w}$ abelian pure Nekrasov-Okounkov (2003)

Outline

I. Random Partitions from Quantum Benjamin-Ono

II. Limit Shapes and Gaussian Fluctuations

III. Integrability of Quantum Benjamin-Ono

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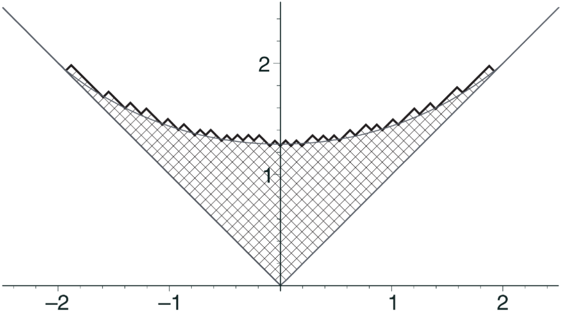
V. A β-Refined Topological Recursion

II. Limit Shapes and Gaussian Fluctuations

For Jack measures $M_v(\varepsilon_2, \varepsilon_1)$, random $\deg(\lambda) = \lambda_1 + \lambda_2 \cdots$

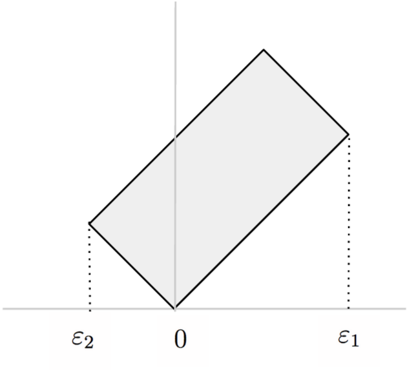
$$\mathbb{E}\big[\deg(\lambda)\big] = \frac{1}{-\varepsilon_1\varepsilon_2} \sum_{k=1}^{\infty} |V_k|^2 < \infty$$

For $v_\bullet(w) = w + \frac{1}{w}$, typical λ from $M_\bullet(-\varepsilon, \varepsilon)$ as $\varepsilon \to 0$:



II. Limit Shapes and Gaussian Fluctuations

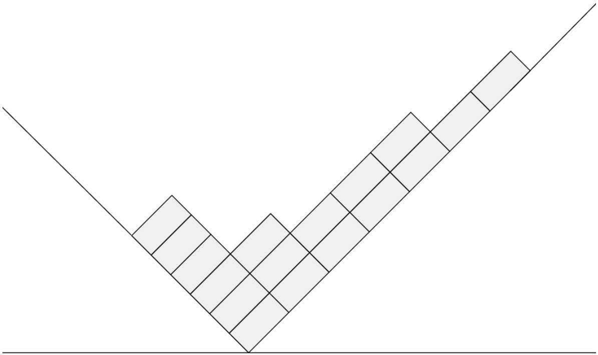
For $\varepsilon_2 < 0 < \varepsilon_1$ real parameters, consider *anisotropic box*



- Area$\left(\square_{\varepsilon_2,\varepsilon_1}\right) = 2(-\varepsilon_1\varepsilon_2)$.
- Isotropic boxes $(\varepsilon_2, \varepsilon_1) \to (-\varepsilon, \varepsilon)$ are squares.

II. Limit Shapes and Gaussian Fluctuations

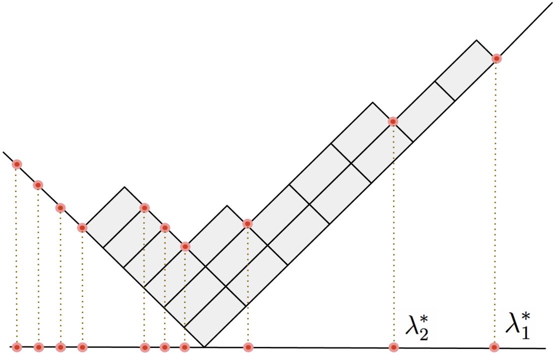
A partition λ ...



Rows: $0 \leq \cdots \leq 0 \leq 1 \leq 1 \leq 1 \leq 2 \leq 5 \leq 7$

II. Limit Shapes and Gaussian Fluctuations

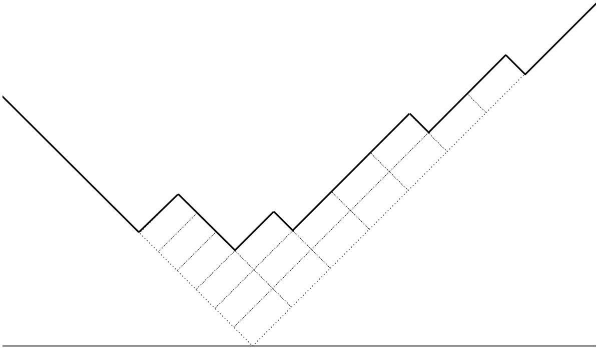
A partition λ ...



Shifted variables: $\lambda_i^* = \varepsilon_2(i-1) + \varepsilon_1 \, \lambda_i$

II. Limit Shapes and Gaussian Fluctuations

A partition λ ... and its anisotropic profile $f_\lambda(c|\varepsilon_2, \varepsilon_1)$



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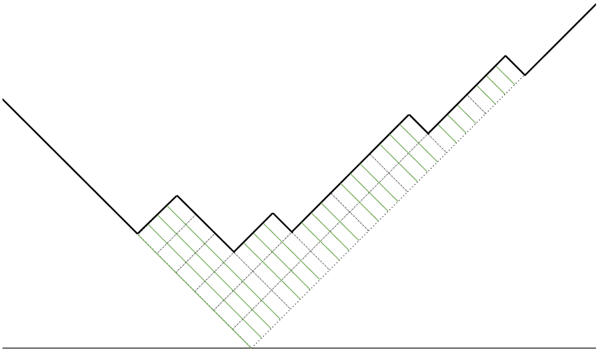

II. Limit Shapes and Gaussian Fluctuations

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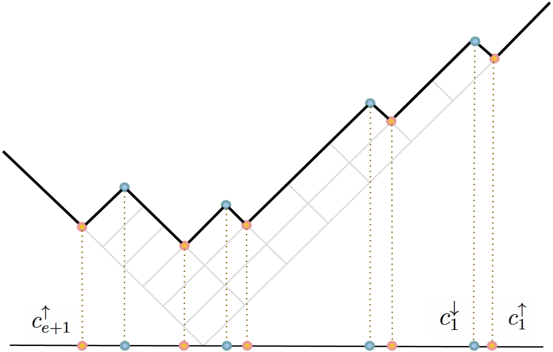


Linear statistics: $\mathrm{ch}_\ell[f] = \int_{-\infty}^{\infty} c^\ell \frac{1}{2} f''(c) dc$

II. Limit Shapes and Gaussian Fluctuations

Partitions: sequences

$$0 \leq \cdots \leq \lambda_2 \leq \lambda_1$$

of non-negative integers $\lambda_i \in \mathbb{N}$ such that

$$\deg(\lambda) := \sum_{i=1}^{\infty} \lambda_i < \infty \qquad\qquad \mathbb{Y} = \bigcup_{d=0}^{\infty} \mathbb{Y}_d$$

Profiles: functions $f : \mathbb{R} \to \mathbb{R}_{\geq 0}$ of $c \in \mathbb{R}$ such that

$$|f(c_1) - f(c_2)| \leq 1 \cdot |c_1 - c_2|$$

$$\mathrm{Area}(f) := \int_{-\infty}^{\infty} (f(c) - |c|) dc < \infty \qquad\qquad \mathcal{Y} = \bigcup_{A=0}^{\infty} \mathcal{Y}(A).$$

II. Limit Shapes and Gaussian Fluctuations

Recall

$$\mathbb{E}[\deg(\lambda)] = \frac{1}{-\varepsilon_1\varepsilon_2} \sum_{k=1}^{\infty} |V_k|^2$$

Macroscopic Scaling: If we *choose* to represent λ as *anisotropic partition* $\lambda \in \mathbb{Y}(\varepsilon_2, \varepsilon_1)$ with the *same* $\varepsilon_2, \varepsilon_1$ defining $M_v(\varepsilon_2, \varepsilon_1)$, then

$$\mathbb{E}\big[\mathrm{Area}(f_\lambda(\cdot|\varepsilon_2, \varepsilon_1))\big] = 2 \sum_{k=1}^{\infty} |V_k|^2$$

is <u>independent of both ε_1 and ε_2!</u>.

Thus, have scaled to "see something" as either $\varepsilon_2 \to 0 \leftarrow \varepsilon_1$.

II. Limit Shapes and Gaussian Fluctuations

For random λ sampled from $M_v(\varepsilon_2, \varepsilon_1)$ with analytic symbol v, in the limit $\varepsilon_2 \to 0 \leftarrow \varepsilon_1$ taken at fixed rate $\beta/2 = -\varepsilon_2/\varepsilon_1 > 0$,

Theorem 1 [M. 2015] *The random profile*

$$f_\lambda(c|\varepsilon_2, \varepsilon_1) \to f_{\star|v}(c)$$

concentrates on ***limit shape*** $f_{\star|v}(c) \in \mathcal{Y}$*, independent of* β*:*

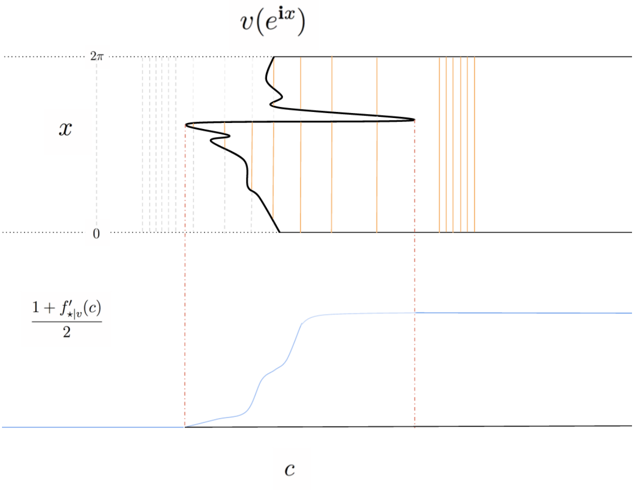
$$2\pi \cdot \frac{1+f'_{\star|v}(c)}{2} = (v_* d\theta)\Big((-\infty, c)\Big)$$

is the distribution function of the **push-forward along** $v : \mathbb{T} \to \mathbb{R}$ of the **uniform measure on the circle**.

- Recover: $\beta = 2$ Okounkov (2003)

II. Limit Shapes and Gaussian Fluctuations

Observe: we are in one-cut regime due to ***regularity*** of v!



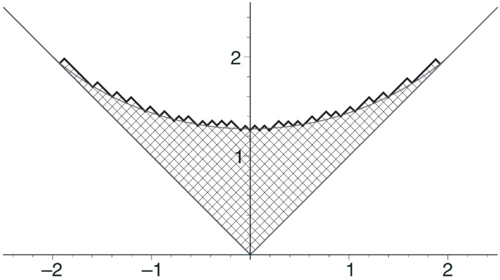
Observe: unlike $f_\lambda(c|\,\varepsilon_2, \varepsilon_1)$, limit shape $f_{\star|v}(c)$ is **convex**!

II. Limit Shapes and Gaussian Fluctuations

Recover: Poissonized Plancherel $v_\bullet(e^{\mathbf{i}\theta}) = 2\cos\theta$, new proof

$$f'_{\star|\bullet}(c) = \tfrac{2}{\pi}\arcsin\tfrac{c}{2}$$

of Vershik-Kerov + Logan-Shepp (1977).



Corrections? Kerov (1993), Ivanov-Olshanski (2003)

II. Limit Shapes and Gaussian Fluctuations

For random λ sampled from $M_v(\varepsilon_2, \varepsilon_1)$ with analytic symbol v, in the limit $\varepsilon_2 \to 0 \leftarrow \varepsilon_1$ taken at fixed rate $\beta/2 = -\varepsilon_2/\varepsilon_1 > 0$,

Theorem 2 [M. 2015] *Profile fluctuations*

$$\phi_\lambda(c|\varepsilon_2, \varepsilon_1) := \frac{1}{\sqrt{-\varepsilon_1\varepsilon_2}}\Big(f_\lambda(c|\varepsilon_2, \varepsilon_1) - f_{\star|v}(c)\Big)$$

converge to a Gaussian field: besides explicit shift $X_v(c)$,

$$\phi_v(c) = (v_* \Phi^{\mathbb{H}_+}\Big|_{\mathbb{T}_+})(c) + \left(\sqrt{\frac{2}{\beta}} - \sqrt{\frac{\beta}{2}}\right) X_v(c).$$

this is **push-forward along** $v : \mathbb{T} \to \mathbb{R}$ **of the restriction to** $\mathbb{T}_+ = \mathbb{T} \cap \mathbb{H}_+$ **of the *Gaussian free field* on** $\mathbb{H}_+$

$$\mathrm{Cov}\Big[\Phi^{\mathbb{H}_+}(w_1), \Phi^{\mathbb{H}_+}(w_2)\Big] = \frac{1}{4\pi} \log \left| \frac{w_1 - \overline{w_2}}{w_1 - w_2} \right|^2$$

with zero boundary conditions.

II. Limit Shapes and Gaussian Fluctuations

Gaussian Free Field at variance $T > 0$ is random

$$\Phi : D \to \mathbb{R}$$

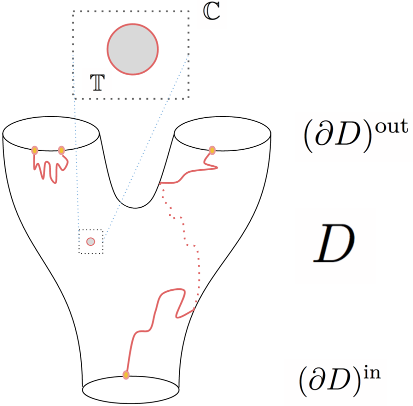
with relative likelihood

$$\exp\left(-\frac{1}{T} \int_D |\nabla \Phi|^2 dz d\overline{z} \right)$$

and boundary conditions

$$\Psi^{\text{out}} \amalg \Psi^{\text{in}}$$

$H^{1/2}(\mathbb{T})$ Pink **Noise**



II. Limit Shapes and Gaussian Fluctuations

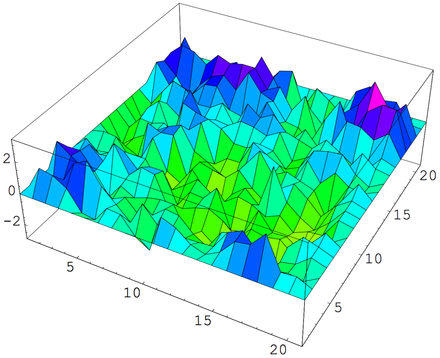
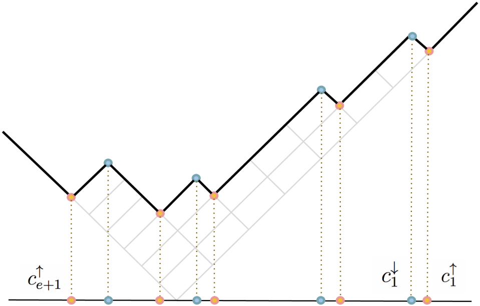


Figure 4.1: Discrete Gaussian free field on 20 by 20 grid with zero boundary conditions.

- Compare: Borodin *Gaussian free field in β-ensembles and random surfaces* (Lecture C.M.I. 2013) and $v(w) = w + \frac{1}{w}$
- Matches: Breuer-Duits' CLT (2013) for Borodin's biorthogonal ensembles at $\beta = 2$ with "symbol" v

II. Limit Shapes and Gaussian Fluctuations: **Why?**

Interlacing measures $\frac{1}{2}f''_\lambda(c|\,\varepsilon_2,\varepsilon_1)$ are **energy densities**

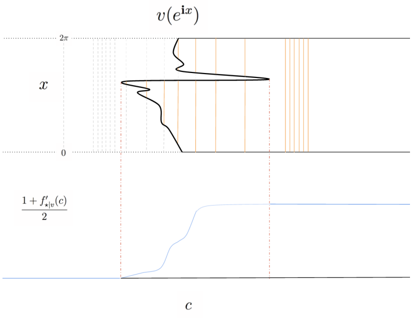


for the *quantum Benjamin-Ono Hamiltonian*

$$\mathcal{Y}_3(\varepsilon_2,\varepsilon_1)\Big|_{P_\lambda(\mathcal{V}|\,\varepsilon_2,\varepsilon_1)} = \int_{-\infty}^{\infty} c^3 \cdot \frac{1}{2}f''_\lambda(c|\,\varepsilon_2,\varepsilon_1)dc$$

II. Limit Shapes and Gaussian Fluctuations: **Why?**

On the other hand, $\frac{1}{2} f''_{\star|v}(c|0,0)$ are **energy densities**



for the *classical Hopf Hamiltonian*

$$\mathcal{Y}_3(0,0)\Big|_{v(w)} := \oint_{\mathbb{T}} v(w)^3 \frac{dw}{2\pi\mathbf{i}w} =: \int_{-\infty}^{\infty} c^3 \cdot \tfrac{1}{2} f''_v(c|0,0)dc$$

II. Limit Shapes and Gaussian Fluctuations: **Why?**

⋆ *Correspondence Principle as Large Deviations Principle?* ⋆

$$
\begin{tikzcd}
\frac{1}{2} f''_{\lambda}(c \,|\, \varepsilon_2, \varepsilon_1) \quad \text{Jack Measures} \arrow[r, dashed] \arrow[d, Leftrightarrow] & \text{Schur Measures} \arrow[d, Leftrightarrow] \\
\Psi^{-\varepsilon_1\varepsilon_2}_{v(w)} \quad \text{Quantum B-O} \quad (\varepsilon_2, \varepsilon_1) \arrow[r, dashed] \arrow[d] \arrow[dr, rightsquigarrow] & (-\varepsilon, \varepsilon) \quad \text{Quantum Hopf} \arrow[d] \\
v(w) \quad \text{Classical B-O} \quad (0, \varepsilon_1) \arrow[r, dashed] & (0,0) \quad \text{Classical Hopf}
\end{tikzcd}
$$

Theorem 1 [M. 2015]: In *dispersionless and classical limit*

$$\varepsilon_2 \to 0 \leftarrow \varepsilon_1$$

the **random** energy density of a **coherent state** $\Psi_{v,-\varepsilon_1\varepsilon_2}$ **around** $\underline{v}$ w.r.t. QBO concentrates on energy density of classical Hopf **at** $\underline{v}$.

Circular β-Ensembles: **N wears three masks**

From Quantum Calogero-Sutherland to $v_t + vv_x = 0$ on $\mathbb{T}$:

$$\begin{tikzcd}
 & (\infty, \varepsilon_2, \varepsilon_1) \arrow[r, dashed] \arrow[dr] & (\infty, -\varepsilon, \varepsilon) \arrow[dr] & \\
(N, \varepsilon_2, \varepsilon_1) \arrow[ur, dotted, "\text{hydro}"] \arrow[r, dashed, "\text{free}"] \arrow[dr, "\text{classical}"'] & (N, -\varepsilon, \varepsilon) \arrow[ur, dotted] \arrow[dr] & (\infty, 0, \varepsilon_1) \arrow[r, dashed] & (\infty, 0, 0) \\
 & (N, 0, \varepsilon_1) \arrow[r, dashed] \arrow[ur, dotted] & (N, 0, 0) \arrow[ur, dotted] &
\end{tikzcd}$$

1. Regularization: require ρ to be N-point configuration
2. Interaction:
$$\varepsilon_1 + \varepsilon_2 = \frac{1}{N}\left(\frac{2}{\beta} - 1\right)$$
3. Quantization:
$$-\varepsilon_1\varepsilon_2 = \frac{1}{N^2} \cdot \frac{2}{\beta}$$

Outline

III. Integrability of Quantum Benjamin-Ono

Quick: At $(\varepsilon_2, \varepsilon_1) = (0, 0)$, the classical Hopf equation $v_t + vv_x = 0$ on $\mathbb{T}$ has an infinite hierarchy

$$\mathcal{Y}_\ell(0,0)\Big|_v := \oint_{\mathbb{T}} v(w)^\ell \frac{dw}{2\pi \mathbf{i} w}$$

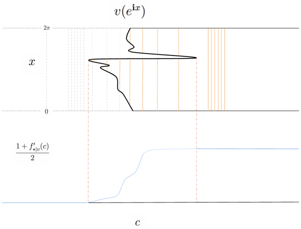
of *local* conserved quantities $\ell = 0, 1, 2, 3, \ldots$

⋆⋆⋆ This does not contradict the gradient catastrophe! ⋆⋆⋆

These are *moments*

$$\mathcal{Y}_\ell(0,0)\Big|_v := \int_{-\infty}^{\infty} c^\ell \tfrac{1}{2} f''_{\star|v}(c|0,0)dc$$

of push-forward along $v : \mathbb{T} \to \mathbb{R}$.



Hard: rewrite $\mathcal{Y}_\ell(0,0)\big|_v$ to generalize to $\mathcal{Y}_\ell(\varepsilon_2, \varepsilon_1)$ for QBO

III. Integrability of Quantum Benjamin-Ono

$$L^2(\mathbb{T}) = H_- \oplus H_0 \oplus H_+$$

with projections π_+, $\pi_\bullet$ onto *Hardy spaces* $H_0 \oplus H_+ =: H_\bullet$.

Form **Toeplitz operators** with symbol $v : \mathbb{T} \to \mathbb{R}$.

$$\begin{aligned} \mathcal{L}_\bullet(0,0)\big|_v &:= \pi_\bullet M(v) \pi_\bullet \\ \mathcal{L}_+(0,0)\big|_v &:= \pi_+ M(v) \pi_+ \end{aligned}$$

Claim: $\mathcal{Y}_\ell(0,0)\big|_v = \oint_{\mathbb{T}} v(w)^\ell \frac{dw}{2\pi \mathbf{i} w}$ is also

$$\int_{-\infty}^{\infty} c^\ell \tfrac{1}{2} f''_{\star|v}(c|0,0) dc = \mathrm{Tr}_{\mathbb{C}[w]}\Big(\mathcal{L}_\bullet^\ell - \mathcal{L}_+^\ell\Big)$$

hence

$$\xi_{\star|v}(c) = \frac{1 + f'_{\star|v}(c)}{2}$$

is the *spectral shift function* ξ for the pair $\mathcal{L}_\bullet(v), \mathcal{L}_+$.

III. Integrability of Quantum Benjamin-Ono

Recall $\mathcal{F} = \bigotimes_{k=1}^{\infty} \mathbb{C}[\mathcal{V}_k]$ with $\mathcal{V}_{-k} := -\varepsilon_1\varepsilon_2\, k \frac{\partial}{\partial \mathcal{V}_k}$. Define

$$\mathbf{v}(w|-\varepsilon_1\varepsilon_2) := \sum_{k=1}^{\infty} (\mathcal{V}_{-k} w^k + \mathcal{V}_k w^{-k})$$

the **Kac-Moody current** for $\widehat{\mathfrak{gl}_1}$.

In an *auxiliary Hardy space* $\mathbb{C}[w]$, define

$$\mathcal{L}_\bullet(\varepsilon_2, \varepsilon_1) : \mathcal{F} \otimes \mathbb{C}[w] \to \mathcal{F} \otimes \mathbb{C}[w]$$

the **Nazarov-Sklyanin Lax operator**

$$\mathcal{L}_\bullet(\varepsilon_2, \varepsilon_1) := \pi_\bullet M(\mathbf{v}(w|-\varepsilon_1\varepsilon_2))\pi_\bullet + (\varepsilon_1 + \varepsilon_2)\mathbb{1} \otimes w\frac{\partial}{\partial w}.$$

Even if $\varepsilon_1 + \varepsilon_2 \to 0$, $\mathcal{L}_\bullet(-\varepsilon, \varepsilon)$ *unbounded* on $\mathcal{F} \otimes \mathbb{C}[w]$.

III. Integrability of Quantum Benjamin-Ono

$\mathcal{L}_\bullet(\varepsilon_2, \varepsilon_1) : \mathbb{C}[w] \to \mathbb{C}[w]$ with coef $\langle h_+|\mathcal{L}|h_-\rangle : \mathcal{F} \to \mathcal{F}$ is

$$\begin{bmatrix} 0 & \mathcal{V}_1 & \mathcal{V}_2 & \mathcal{V}_3 & \cdots & \mathcal{V}_h & \cdots \\ \mathcal{V}_{-1} & (\varepsilon_1+\varepsilon_2) & \mathcal{V}_1 & \mathcal{V}_2 & \ddots & \mathcal{V}_{h-1} & \ddots \\ \mathcal{V}_{-2} & \mathcal{V}_{-1} & \mathbf{2}(\varepsilon_1+\varepsilon_2) & \mathcal{V}_1 & \ddots & \ddots & \ddots \\ \mathcal{V}_{-3} & \mathcal{V}_{-2} & \mathcal{V}_{-1} & \mathbf{3}(\varepsilon_1+\varepsilon_2) & \ddots & \ddots & \ddots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\ \mathcal{V}_{-h} & \mathcal{V}_{-(h-1)} & \ddots & \ddots & \ddots & \mathbf{h}(\varepsilon_1+\varepsilon_2) & \ddots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \end{bmatrix}$$

- (pairings) When $k_+ = k_- = \mathbf{k}$, $[\mathcal{V}_{-k}, \mathcal{V}_k] = -\varepsilon_1\varepsilon_2\mathbf{k}$
- (slides) When $h_+ = h_- = \mathbf{h}$, $\mathcal{L}_{h,h} = (\varepsilon_1+\varepsilon_2)\mathbf{h}$.
- $\mathcal{L}_{h,h} \equiv 0$ if $\varepsilon_1 + \varepsilon_2 = 0$ (*no slides iff* $\beta = 2$).

III. Integrability of Quantum Benjamin-Ono

Theorem [N.S. 2013]: For $\ell = 0, 1, 2, 3, \dots$ the operators

$$\mathcal{Y}_\ell(\varepsilon_2, \varepsilon_1) := \mathrm{Tr}_{\mathbb{C}[w]}\left(\mathcal{L}_\bullet^\ell - \mathcal{L}_+^\ell\right)$$

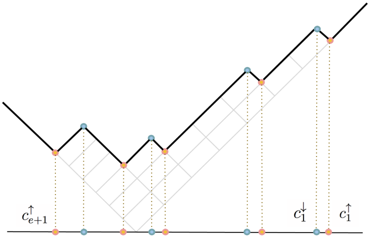
are well-defined, self-adjoint, commuting operators on $\mathcal{F}$ simultaneously diagonalized on Jacks with eigenvalue

$$\mathcal{Y}_\ell(\varepsilon_2, \varepsilon_1)\Big|_{P_\lambda(\mathcal{V}|\,\varepsilon_2, \varepsilon_1)} = \int_{-\infty}^{\infty} c^\ell \cdot \tfrac{1}{2} f_\lambda''(c|\,\varepsilon_2, \varepsilon_1)\,dc$$

The energy density is the interlacing measure

$$\tfrac{1}{2} f_\lambda''(c|\,\varepsilon_2, \varepsilon_1)$$

and depends only on the profile of $(\lambda, \varepsilon_2, \varepsilon_1)$.



III. Integrability of Quantum Benjamin-Ono

Proof? ⋆ **RTT = TTR?** ⋆

- Trigonometric Dunkl-Cherednik (1991) for dDAHA(N)

$$D_i^{(\beta)} = D_i - \tfrac{\beta}{2}\widetilde{D}_i$$

- Sekiguchi (1977), Debiard (1983)

$$\det S_N(u|\beta) = \prod_{i=1}^{N}(u - D_i^{(\beta)})$$

- Nazarov-Sklyanin (2013) stable $\det S_N(u|\beta)$ as $N \to \infty$ and

$$\mathcal{Y}(u|\beta) = \frac{\partial}{\partial u}\log\left[\frac{\det S_\infty(u|\beta)}{\det S_\infty(u-1|\beta)}\right]$$

- Compare: Bernard-Gaudin-Haldane-Pasquier (1993)

$$\mathrm{qDet}^*\,\widehat{T}_N(u|\beta) = \frac{\det S_N(u|\beta)}{\det S_N(u-1|\beta)}$$

III. Integrability of Quantum Benjamin-Ono

⋆⋆ Kerov's Markov-Krein Correspondence ⋆⋆

$$\mathcal{Y}(u|\,\varepsilon_2,\varepsilon_1) := \frac{\partial}{\partial u}\log \mathcal{T}^{\uparrow}(u|\,\varepsilon_2,\varepsilon_1)$$

Compare:

- Biane's Jucy-Murphy elements of $S(d)$ (1998)
- Eliashburg (2006), Dubrovin (2015) for Q-Hopf
- Schiffmann-Vasserot (2011) β-deformation of $W_{1+\infty}$
- Maulik-Okounkov (2012) affine Yangian $\mathrm{Y}_{\varepsilon_1+\varepsilon_2}(\widehat{\mathfrak{gl}_1})$

$$\mathbf{T}(u|\,\varepsilon_2,\varepsilon_1) = \langle 0|\mathbf{R}(u|\,\varepsilon_2,\varepsilon_1)|0\rangle$$

Recover: abelian pure $\mathcal{Y}(u|\,\varepsilon_2,\varepsilon_1)$ Nekrasov (2016)

III. Integrability of Quantum Benjamin-Ono: **RMT!**

$\mathcal{L}_\bullet(\varepsilon_2, \varepsilon_1) \Rightarrow \mathbb{L}_\bullet(\varepsilon_2, \varepsilon_1)\big|_v$ for ***macroscopics*** of Jack Meas.

$$\begin{bmatrix} \mathbf{0} & \mathbb{V}_1 & \mathbb{V}_2 & \mathbb{V}_3 & \cdots & \mathbb{V}_h & \cdots \\ \overline{\mathbb{V}_1} & \mathbf{1}(\varepsilon_1+\varepsilon_2) & \mathbb{V}_1 & \mathbb{V}_2 & \ddots & \mathbb{V}_{h-1} & \ddots \\ \overline{\mathbb{V}_2} & \overline{\mathbb{V}_1} & \mathbf{2}(\varepsilon_1+\varepsilon_2) & \mathbb{V}_1 & \ddots & \ddots & \ddots \\ \overline{\mathbb{V}_3} & \overline{\mathbb{V}_2} & \overline{\mathbb{V}_1} & \mathbf{3}(\varepsilon_1+\varepsilon_2) & \ddots & \ddots & \ddots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\ \overline{\mathbb{V}_h} & \overline{\mathbb{V}_{h-1}} & \ddots & \ddots & \ddots & \mathbf{h}(\varepsilon_1+\varepsilon_2) & \ddots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \end{bmatrix}$$

$\mathbb{V}_k$ are ***independent*** *complex Gaussian random variables*

- $\mathbb{V}_k$ has mean $V_k \in \mathbb{C}$ given by $v(w) = \sum_{k=1}^{\infty} V_k w^k + \overline{V_k} w^{-k}$
- $\mathbb{V}_k$ has complex variance $-\varepsilon_1\varepsilon_2\,\mathbf{k}$ and is circularly symmetric

Compare: $Y^{\pm} \to \mathcal{K}_v$ for ***microscopics*** of Schur measures

III. Integrability of Quantum Benjamin-Ono: **RMT!**

$\mathcal{L}_\bullet(\varepsilon_2,\varepsilon_1) \Rightarrow \mathbb{L}_\bullet(\varepsilon_2,\varepsilon_1)\big|_v$ for ***macroscopics*** of Jack Meas.

$$\begin{bmatrix} \mathbf{0} & \mathbb{V}_1 & \mathbb{V}_2 & \mathbb{V}_3 & \cdots & \mathbb{V}_h & \cdots \\ \overline{\mathbb{V}_1} & \mathbf{1}(\varepsilon_1+\varepsilon_2) & \mathbb{V}_1 & \mathbb{V}_2 & \ddots & \mathbb{V}_{h-1} & \ddots \\ \overline{\mathbb{V}_2} & \overline{\mathbb{V}_1} & \mathbf{2}(\varepsilon_1+\varepsilon_2) & \mathbb{V}_1 & \ddots & \ddots & \ddots \\ \overline{\mathbb{V}_3} & \overline{\mathbb{V}_2} & \overline{\mathbb{V}_1} & \mathbf{3}(\varepsilon_1+\varepsilon_2) & \ddots & \ddots & \ddots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\ \overline{\mathbb{V}_h} & \overline{\mathbb{V}_{h-1}} & \ddots & \ddots & \ddots & \mathbf{h}(\varepsilon_1+\varepsilon_2) & \ddots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \end{bmatrix}$$

Theorem [M. 2015] For $\ell = 0, 1, 2, \ldots$ joint equality in law

$$\int_{-\infty}^{\infty} c^\ell \tfrac{1}{2} f_\lambda''(c|\,\varepsilon_2,\varepsilon_1)dc = \mathrm{Tr}_{\mathbb{C}[w]}\left[\mathbb{L}_\bullet(\varepsilon_2,\varepsilon_1)\big|_v^\ell - \mathbb{L}_+(\varepsilon_2,\varepsilon_1)^\ell\big|_v\right]$$

Random spectral shift fn. of $\mathbb{L}_\bullet(\varepsilon_2,\varepsilon_1)\big|_v$ is *indistinguishable from* random interlacing measure $\frac{1}{2}f_\lambda''(c|\,\varepsilon_2,\varepsilon_1)dc$.

Outline

I. Random Partitions from Quantum Benjamin-Ono

II. Limit Shapes and Gaussian Fluctuations

III. Integrability of Quantum Benjamin-Ono

IV. Poisson at High Frequency

V. A β-Refined Topological Recursion

IV. Poisson at High Frequency

Data	QHO	QBO
$\mathcal{M}$	$V \in \mathbb{C}$	$v(w) = \sum_k V_k w^k$
$\mathcal{F}$	$\mathbb{C}[\mathcal{V}_+]$	$\bigotimes_{k=1}^{\infty} \mathbb{C}[\mathcal{V}_k]$
$\Psi_\circ$	$\|0\rangle$	$\bigotimes_{k=1}^{\infty} \|0\rangle_k$
$\langle\cdot,\cdot\rangle_\varepsilon$	$\mathcal{V}_- := \varepsilon^2 \frac{\partial}{\partial \mathcal{V}_+}$	$\mathcal{V}_{-k} := -\varepsilon_1\varepsilon_2\, k \frac{\partial}{\partial \mathcal{V}_k}$
Hamiltonian	$\mathcal{V}_+\mathcal{V}_-$	$\mathcal{Y}_3(\varepsilon_2, \varepsilon_1)$
Eigenfunctions	$\mathcal{V}_+^d$	$P_\lambda(\mathcal{V}\| \varepsilon_2, \varepsilon_1)$
Coherent State Ψ_V	$\exp\left(\frac{V_k \mathcal{V}_k}{\varepsilon}\right)\|0\rangle$	$\bigotimes_{k=1}^{\infty} \exp\left(\frac{V_k \mathcal{V}_k}{\sqrt{-\varepsilon_1\varepsilon_2}}\right)\|0\rangle_k$
Random$(\lambda) \propto \|\widehat{\Psi_V}(\lambda)\|^2$	Poisson	Jack Measure
Matrix Model	Gaussian $\mathbb{V}$	$\mathbb{L}_V(\varepsilon_2, \varepsilon_1)$

IV. Poisson at High Frequency

For $V \in \mathbb{C}$ and $\varepsilon > 0$, Poisson random $d = 0, 1, 2, 3, \ldots$

$$\mathrm{Prob}_{V,\varepsilon}(d) = \frac{1}{\Pi(V|\,\varepsilon)} \frac{1}{d!} \left(\frac{V\overline{V}}{\varepsilon^2} \right)^d$$

where $\Pi(V|\,\varepsilon) := \exp\left(\frac{V\overline{V}}{\varepsilon^2}\right)$. For $f_d(\cdot|\,\varepsilon) := \varepsilon^2\, d$, equality in law

$$\mathbb{E}_{\text{Poisson}}\left[f_d(\cdot|\,\varepsilon)^n\right] = \mathbb{E}_{\text{Gaussian}}\left[(\mathbb{V}\overline{\mathbb{V}})^n\right]$$

$\mathbb{V}$ complex Gaussian mean V, complex variance ε^2, $\bigcirc$-symmetric.

Theorems: As $\varepsilon \to 0$,

$$f_d(\cdot|\,\varepsilon) \sim f_{\star|V}(\cdot) + \tfrac{1}{\varepsilon}\xi_V(\cdot)$$

1. *Law of Large Numbers:* limit shape $|V|^2$, since pairings cost ε^2 and **to leading order no pairings occur.**
2. *Central Limit Theorem:* Gaussian fluctuations variance $|V|^2$, since **it takes** $n-1$ **pairings to connect** n **vertices.**

Thank You!

Outline

I. Random Partitions from Quantum Benjamin-Ono

II. Limit Shapes and Gaussian Fluctuations

III. Integrability of Quantum Benjamin-Ono

IV. Poisson at High Frequency

V. A β-Refined Topological Recursion

V. A β-Refined Topological Recursion: **Ingredients**

1. Random partition $\lambda \in \mathbb{Y}$ sampled with relative weight

$$W_\lambda(V|\varepsilon) := \left.\frac{P_\lambda(\overline{V^{\text{out}}}|\,\varepsilon_2, \varepsilon_1)P_\lambda(V^{\text{in}}|\,\varepsilon_2, \varepsilon_1)}{\langle P_\lambda, P_\lambda\rangle_{-\varepsilon_1\varepsilon_2}}\right|_{V^{\text{out}}=V^{\text{in}}}$$

2. Partition function: decoupled (emergent GFF)

$$\prod_{k=1}^{\infty} \exp\left(\frac{\overline{V_k^{\text{out}}}V_k^{\text{in}}}{-\varepsilon_1\varepsilon_2\, k}\right) = \Pi = \sum_{\lambda\in\mathbb{Y}} W_\lambda(V|\varepsilon)$$

3. Observables: $\mathcal{L} : \mathcal{F} \otimes \mathbb{C}[w] \to \mathcal{F} \otimes \mathbb{C}[w]$ resolvent VEV

$$\frac{\partial}{\partial u} \log \langle 0| \frac{1}{(u - \mathcal{L}(\varepsilon_2, \varepsilon_1))}|0\rangle P_\lambda = \left(\int_{-\infty}^{\infty} \frac{\frac{1}{2}f_\lambda''(c|\,\varepsilon_2, \varepsilon_1)}{u-c}\right)P_\lambda$$

- Joint moments of linear statistics computable via

$$\Pi^{-1}\langle 0|(u_1 - \mathcal{L}^{\text{out}})^{-1}|0\rangle \cdots \langle 0|(u_n - \mathcal{L}^{\text{out}})^{-1}|0\rangle \Pi$$

V. A β-Refined Topological Recursion: **Cooking**

$$\Pi^{-1}\langle 0|(u_1-\mathcal{L}^{\mathrm{out}})^{-1}|0\rangle\cdots\langle 0|(u_n-\mathcal{L}^{\mathrm{out}})^{-1}|0\rangle\Pi$$

1. $\mathcal{L}(\varepsilon_2,\varepsilon_1)=\mathsf{T}(\mathbf{v}|-\varepsilon_1\varepsilon_2)+(\varepsilon_1+\varepsilon_2)\mathcal{D}$ made of V_k and
$$V_{-k}=-\varepsilon_1\varepsilon_2 k\frac{\partial}{\partial V_k}.$$
2. Keep track of pairings (Leibniz rule) and slides
3. Exchange relation
$$[\overline{V_{-k}^{\mathrm{out}}},\Pi]=V_k^{\mathrm{in}}.$$

from Kac-Moody symbol to scalar symbol $v:\mathbb{T}\to\mathbb{R}$

$$\begin{aligned}\mathbf{v}^{\mathrm{out}}(w|-\varepsilon_1\varepsilon_2) &= \sum_{k=1}^{\infty}\overline{V_k^{\mathrm{out}}}w^{-k}+\sum_{k=1}^{\infty}\overline{V_{-k}^{\mathrm{out}}}w^k\\ v(w) &= \sum_{k=1}^{\infty}\overline{V_k^{\mathrm{out}}}w^{-k}+\sum_{k=1}^{\infty}V_k^{\mathrm{in}}w^k.\end{aligned}$$

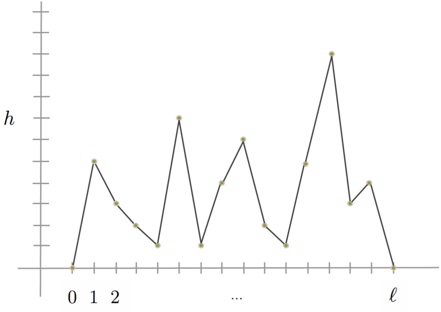
V. A β-Refined Topological Recursion: **Symbols**

At $\varepsilon_2 = 0 = \varepsilon_1$, $n = 1$ case reduces to

$$\Pi^{-1}\langle 0|\mathcal{L}(\varepsilon_2, \varepsilon_1)^{\ell}|0\rangle\Pi \longrightarrow \langle 0|\, T(v)^{\ell}|0\rangle$$

VEV of power of Toeplitz operator with symbol

$$v(w) = \sum_{k=1}^{\infty} \overline{V_k^{\text{out}}} w^{-k} + \sum_{k=1}^{\infty} V_k^{\text{in}} w^k.$$

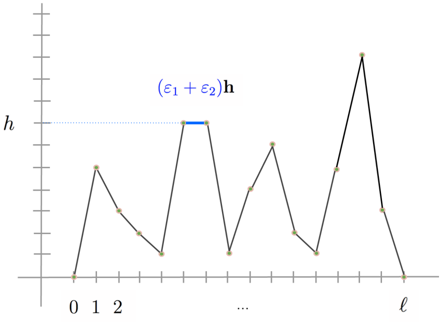


V. A β-Refined Topological Recursion: **Slides**

If $-\varepsilon_1\varepsilon_2 = 0$ but $\varepsilon = \varepsilon_1 + \varepsilon_2 \neq 0$, $n = 1$ case reduces to

$$\Pi^{-1}\langle 0|\mathcal{L}(\varepsilon_2, \varepsilon_1)^{\ell}|0\rangle\Pi \longrightarrow \langle 0|\big(T(v) + \varepsilon\mathcal{D}_{\mathrm{aux}}\big)^{\ell}|0\rangle$$

VEV of power of unbounded perturbation of $T(v)$.



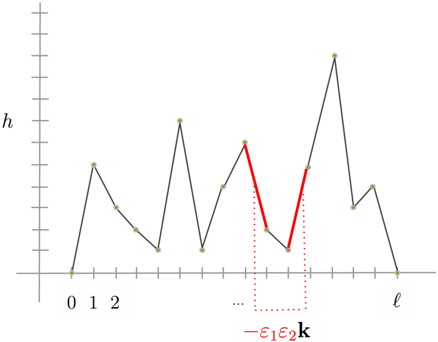
V. A β-Refined Topological Recursion: **Pairings**

If $-\varepsilon_1\varepsilon_2 \neq 0$ but $\varepsilon_1 + \varepsilon_2 = 0$, $n = 1$ case reduces to

$$\Pi^{-1}\langle 0|\mathcal{L}(\varepsilon_2, \varepsilon_1)^{\ell}|0\rangle\Pi \longrightarrow \langle 0|\,T(\mathbf{v}|-\varepsilon_1\varepsilon_2)^{\ell}|0\rangle$$

VEV of Toeplitz operator with $\widehat{\mathfrak{gl}_1}$ current

$$\mathbf{v}(w|-\varepsilon_1\varepsilon_2) = \sum_{k=-\infty}^{\infty} V_k \otimes w^{-k}$$



V. A β-Refined Topological Recursion: **Regularity**

Estimates: although the partition function

$$\Pi = \prod_{k=1}^{\infty} \exp\left(\frac{\overline{V_k^{\mathrm{out}}}\, V_k^{\mathrm{in}}}{-\varepsilon_1\varepsilon_2\, k}\right)$$

is convergent if the potential

$$V(w) = \sum_{k=1}^{\infty} \frac{\overline{V_k^{\mathrm{out}}}\, w^{-k}}{k} + \sum_{k=1}^{\infty} \frac{V_k^{\mathrm{in}} w^k}{k}$$

lies in $H^{1/2}(\mathbb{T})$, our operators $\langle 0|\mathcal{L}^{\ell}|0\rangle$ only defined on $\mathcal{F}$.

<ins>Lemma:</ins> Π is in the domain of definition of $\langle 0|\mathcal{L}^{\ell}|0\rangle$ iff

$$V \in H^s(\mathbb{T}) \qquad \text{for} \quad \ell = 2s.$$

In particular, moment method requires all Sobolev norms.

- $V : \mathbb{T} \to \mathbb{C}$ analytic around $\mathbb{T}$ lies in all $H^s(\mathbb{T})$

V. A β-Refined Topological Recursion: **Expansions**

For random λ sampled from $M_V(\varepsilon_2, \varepsilon_1)$ with *analytic* symbol v,

Theorem 3 [M. 2015]: $\widehat{W}^v_n(\ell_1, \ldots, \ell_n | \varepsilon_1, \varepsilon_2)$ joint cumulants of $\langle 0 | \mathcal{L}^{\ell_i} | 0 \rangle$ have *convergent* expansion:

$$\sum_{g=0}^{\infty} \sum_{m=0}^{\infty} (-\varepsilon_1 \varepsilon_2)^{(\mathbf{n}-1)+g} (\varepsilon_1 + \varepsilon_2)^m \widehat{W}^v_{n,g,m}(\ell_1, \ldots, \ell_n)$$

$\widehat{W}^v_{n,g,m}(\ell_1, \ldots, \ell_n)$ weighted enumeration of **connected "ribbon paths"** on n sites of lengths $\ell_1, \ldots, \ell_n$ with $(n-1)+g$ *pairings* and m *slides*, and are expressed solely through ***matrix elements of the classical Toeplitz operator*** $T(v)$ ***with scalar symbol*** $v(w)$**.**

- Recover: double Hurwitz numbers for Schur measures at $\beta = 2$ or $\varepsilon_1 + \varepsilon_2 = 0$ Okounkov (2000)
- Compare: loop equations and refined topological recursion Chekhov-Eynard (2006), Borot-Guionnet (2012)

V. A β-Refined Topological Recursion: **Expansions**

Compare: expansion of β-Ensembles on $\mathbb{R}$ for one-cut V:

Theorem [C-E + B-G]: After the change of variables

$$-\varepsilon_1\varepsilon_2 = \frac{2}{\beta}\cdot\frac{t^2}{N^2} \qquad \varepsilon_1+\varepsilon_2 = \left(\frac{2}{\beta}-1\right)\frac{t}{N}.$$

the joint cumulants $\widehat{W}_n^V(\ell_1,\ldots,\ell_n|\varepsilon_2,\varepsilon_1)$ of linear statistics $\int_{-\infty}^{\infty} x^\ell \rho(x)dx$ have *asymptotic* expansion:

$$\sim \sum_{g=0}^{\infty}\sum_{m=0}^{\infty}(-\varepsilon_1\varepsilon_2)^{(\mathbf{n}-1)+g}(\varepsilon_1+\varepsilon_2)^m\,\widehat{W}_{n,g,m}^{v}(\ell_1,\ldots,\ell_n)$$

where $\widehat{W}_{n,g,m}^V(\ell_1,\ldots,\ell_n)$ is V-weighted enumeration of **connected ribbon graphs** on n vertices of degree $\ell_1,\ldots,\ell_n$ of genus g and m *Möbius strips*, and are expressed solely through ***geometry of the spectral curve*** Σ_V**.**

V. A β-Refined Topological Recursion: **Expansions**

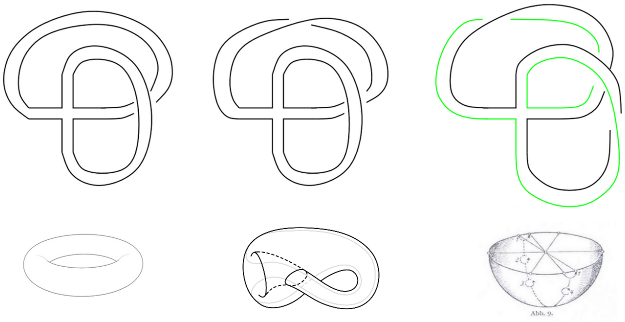


Figure 7: Examples of ribbon graphs of genus $g = 1$.

- Source: Chekhov-Eynard-Marchal *Topological expansion of the Bethe ansatz and quantum algebraic geometry* (2009)
- Compare: $-\varepsilon_1\varepsilon_2$ as "handle-gluing element" in Maulik-Okounkov (2012)

V. A β-Refined Topological Recursion: **Limit Shapes**

Proof Sketch LLN:

- *LLN occurs:* no pairings or slides to leading order.
- *LLN form:* to determine limit shape $f_{\star|v}(c)$, must invert

$$W^{v}_{1,0,0}(u) := \sum_{\ell=0}^{\infty} u^{-\ell-1}\widehat{W}^{v}_{1,0,0}(\ell)$$

which requires the analytic continuation

$$\langle 0|(u - T(v))^{-1}|0\rangle = \exp\left(\frac{1}{2\pi\mathbf{i}}\oint_{\mathbb{T}} \log\left[\frac{1}{u - v(w)}\right]\frac{dw}{w}\right)$$

Krein (1958) and Calderón-Spitzer-Widom (1958), via Wiener-Hopf factorization of the *family of loops*

$$\gamma(w;u) : \mathbb{T} \to GL(1) \qquad\qquad \gamma(w;u) = u - v(w)$$

- *LLN meaning*: Lifshitz-Krein spectral shift function.

V. A β-Refined Topological Recursion: GFF

Proof Sketch CLT:

- *CLT occurs:* it requires $n-1$ pairings to connect n sites, and slides cannot connect correlators
- *CLT form:* to compute covariance of limiting Gaussian field, need $W^{v}_{2,0,0}(u_1,u_2)$: introduce **welding operator**

$$\mathcal{W} := \sum_{k=1}^{\infty} k \frac{\partial}{\partial V^{(1)}_{-k}} \frac{\partial}{\partial V^{(2)}_{k}} \Bigg|_{v^{(1)}=v^{(2)}}.$$

  Duplicate variables, simulates all ways of creating one pairing of any type k, hence

$$\mathcal{W} : W^{v^{(1)}}_{1,0,0}(u_1) \times W^{v^{(2)}}_{1,0,0}(u_2) \to W^{v}_{2,0,0}(u_1,u_2)$$

  have $LLN \times LLN \Rightarrow CLT$ covariance.
- Compare: **loop insertion operator** $\mathcal{K}(u)$ and **unstable correlators** $W^{V}_{1,0,0}(u)$, $W^{V}_{1,0,1}(u)$, $W^{V}_{2,0,0}(u_1,u_2)$