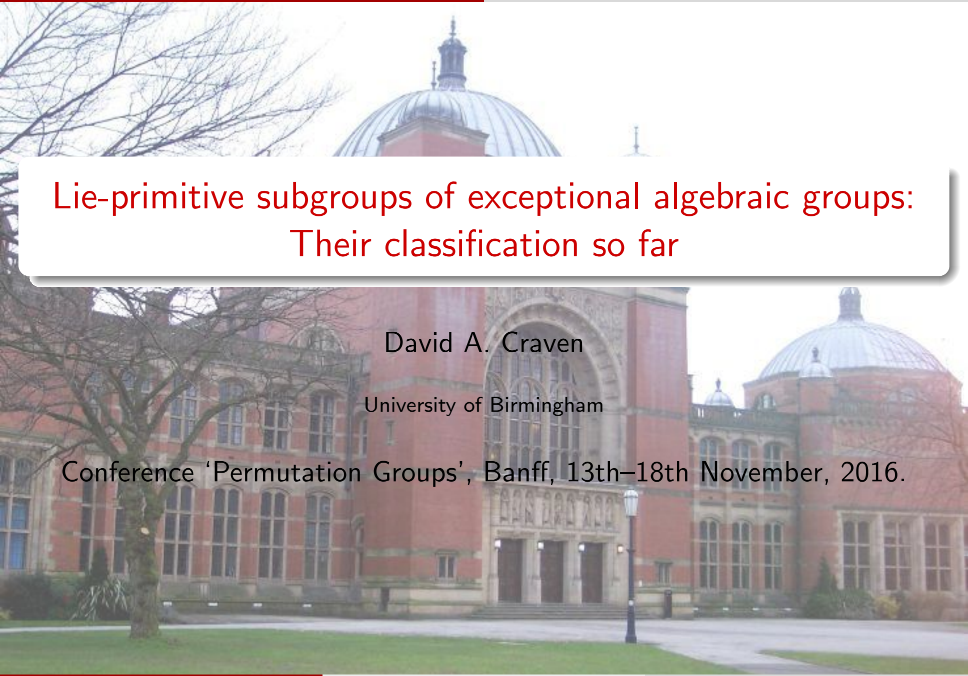

Lie-primitive subgroups of exceptional algebraic groups: Their classification so far

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Since all maximal closed, positive-dimensional subgroups of G are known (see Liebeck–Seitz), we can (at least in theory) find all imprimitive subgroups of G. Thus Lie-primitive subgroups are the main impediment to understanding all finite subgroups of G, in particular the maximal subgroups of the finite exceptional groups of Lie type.

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Theorem

Suppose that $H = H(q)$ is a Lie-primitive simple subgroup of the exceptional algebraic group G, where q is a power of $p = \mathrm{char}(G)$. Then the untwisted rank of H (so 4 for ${}^3D_4$, for example) is at most half of that of G, and one of the following holds:

- $q \leq 9$,
- $H = \mathrm{PSL}_3(16), \mathrm{PSU}_3(16)$,
- $H = \mathrm{PSL}_2(q)$, ${}^2B_2(q)$ *or* ${}^2G_2(q)$, *and* $q < \mathrm{gcd}(2, p) \cdot t(G)$, *where* $t(G)$ *is as follows:*

G	G_2	F_4	E_6	E_7	E_8
$t(G)$	12	68	124	388	1312

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Disclaimer: Many papers have been written on this problem, and **I think** I am correct on the current state of knowledge. I would be very happy to be corrected.

F_4: non-equicharacteristic case

All simple subgroups of F_4:

p	H
All primes	Alt_{5-6}, $\mathrm{PSL}_2(q)$, $q = 7, 8, 13, 17, 25, 27$, $\mathrm{PSL}_3(3)$, $\mathrm{PSU}_3(3)$, ${}^3D_4(2)$
$p = 2$	$\mathrm{Alt}_{7,9,10}$, J_2, $\mathrm{PSL}_4(3)$
$p = 3$	$\mathrm{PSL}_3(4)$
$p = 5$	Alt_7
$p = 11$	J_1, M_{11}

F_4: non-equicharacteristic case

After Cohen–Wales, Litterick and Magaard:

p	H
$p \nmid \lvert H \rvert$	$\mathrm{PSL}_2(q)$, $q = 9, 25, 27$
$p = 2$	Alt_7, $\mathrm{PSL}_2(q)$, $q = 13, 25, 27$, $\mathrm{PSL}_3(3)$, $\mathrm{PSL}_4(3)$
$p = 3$	$\mathrm{PSL}_2(q)$, $q = 7, 8, 13, 17, 25$, ${}^3D_4(2)$, $\mathrm{PSL}_3(4)$
$p = 5$	Alt_6
$p = 7$	$\mathrm{PSL}_2(q)$, $q = 8, 25, 27$, $\mathrm{PSU}_3(3)$, ${}^3D_4(2)$
$p = 13$	$\mathrm{PSL}_2(q)$, $q = 25, 27$

F_4: all cases

Current state:

p	H
$p \nmid \lvert H \rvert$	$\mathrm{PSL}_2(q)$, $q = 9, 25, 27$
$p = 2$	$\mathrm{PSL}_2(q)$, $q = 13, 25$, $\mathrm{PSL}_3(3)$
$p = 3$	$\mathrm{PSL}_2(q)$, $q = 9, 13, 25$
$p = 7$	$\mathrm{PSL}_2(q)$, $q = 8, 13, 27$
$p = 13$	$\mathrm{PSL}_2(q)$, $q = 25, 27$

(The case $\mathrm{PSL}_2(27)$ for characteristic 2 was proved by Magaard and Parker, and the case $\mathrm{PSL}_2(13)$ for characteristic 13 was proved by Burness and Testerman.)

E_6: non-equicharacteristic case

All simple subgroups of E_6:

p	H
All primes	Alt_{5-7}, M_{11}, $\mathrm{PSL}_2(q)$, $q = 7, 8, 11, 13, 17, 19, 25, 27$, $\mathrm{PSL}_3(3)$, $\mathrm{PSU}_3(3)$, $\mathrm{PSU}_4(2)$, ${}^3D_4(2)$, ${}^2F_4(2)'$
$p = 2$	Alt_{9-12}, M_{12}, M_{22}, J_2, J_3, Fi_{22}, $\mathrm{PSL}_4(3)$, $\mathrm{PSU}_4(3)$, $\Omega_7(3)$, $G_2(3)$
$p = 5$	M_{12}
$p = 11$	J_1

E_6: all cases

After Cohen–Wales, Litterick and particularly Aschbacher:

p	H
$p \nmid \lvert H \rvert$	$\mathrm{PSL}_2(q)$, $q = 7, 9, 19$, $\mathrm{PSL}_3(3)$, $\mathrm{PSU}_3(3)$,
$p = 2$	J_2, Alt_8, $\mathrm{PSL}_2(q)$, $q = 9, 13, 19$,
$p = 3$	$\mathrm{PSL}_2(q)$, $q = 13, 19$
$p = 5$	$\mathrm{Alt}_{6,7}$, $\mathrm{PSL}_2(19)$, M_{11}, M_{12}
$p = 11$	J_1
$p = 13$	$\mathrm{PSL}_3(3)$

E_6: all cases

Current state:

p	H
$p \nmid \lvert H\rvert$	$\mathrm{PSL}_2(q)$, $q = 7, 9, 19$, $\mathrm{PSL}_3(3)$
$p = 2$	$\mathrm{PSL}_2(q)$, $q = 13, 19$
$p = 3$	$\mathrm{PSL}_2(q)$, $q = 13, 19$
$p = 5$	$\mathrm{PSL}_2(19)$
$p = 13$	$\mathrm{PSL}_3(3)$

E_7: non-equicharacteristic case

All simple subgroups of E_7:

p	H
All primes	Alt_{5-9}, $\mathrm{PSL}_2(q)$, $q = 7, 8, 11, 13, 17, 19, 25, 27, 29, 37$, $\mathrm{PSL}_3(3)$, $\mathrm{PSL}_3(4)$, $\mathrm{PSU}_3(3)$, $\mathrm{PSU}_3(8)$, $\mathrm{PSU}_4(2)$, $\mathrm{Sp}_6(2)$, $\Omega_8^+(2)$, ${}^3D_4(2)$, ${}^2F_4(2)'$, M_{11}, M_{12}, J_2
$p = 2$	Alt_{10-13}, $\mathrm{PSL}_4(3)$
$p = 5$	Alt_{10}, M_{22}, Ru, HS
$p = 11$	J_1

E_7: non-equicharacteristic case

After Litterick:

p	H
$p \nmid \lvert H\rvert$	$\mathrm{PSL}_2(q)$, $q = 5, 7, 9, 11, 13, 19, 27, 29, 37$, $\mathrm{PSL}_3(4)$, $\mathrm{PSU}_3(3)$, $\mathrm{PSU}_3(8)$, $\Omega_8^+(2)$,
$p = 2$	J_2, $\mathrm{PSL}_2(q)$, $q = 11, 13, 19, 27, 29, 37$
$p = 3$	$\mathrm{PSL}_2(q)$, $q = 7, 8, 11, 13, 19, 27, 29, 37$, $\mathrm{PSL}_3(4)$, $\mathrm{PSU}_3(8)$, $\Omega_8^+(2)$, $\mathrm{Alt}_{8,9}$
$p = 5$	M_{12}, M_{22}, Ru, HS, $\mathrm{Alt}_{7,8}$, $\mathrm{PSL}_2(q)$, $q = 5, 9, 11, 19, 29$, $\mathrm{PSL}_3(4)$, $\Omega_8^+(2)$, ${}^2B_2(8)$
$p = 7$	$\mathrm{PSL}_2(q)$, $q = 8, 13, 27, 29$, $\mathrm{PSL}_3(4)$, $\mathrm{PSU}_3(8)$, $\Omega_8^+(2)$
$p = 13$	$\mathrm{PSL}_2(27)$
$p = 19$	$\mathrm{PSL}_2(37)$, $\mathrm{PSU}_3(8)$

E_7: all cases

Current state:

p	H
$p \nmid \vert H\vert$	$\mathrm{PSL}_2(q)$, $q = 7, 9, 11, 13, 19, 27, 29, 37$, $\mathrm{PSL}_3(4)$, $\mathrm{PSU}_3(3)$
$p = 2$	$\mathrm{PSL}_2(q)$, $q = 8, 19, 27, 29, 37$, $\mathrm{PSL}_3(4)$, $\mathrm{PSU}_3(3)$
$p = 3$	$\mathrm{PSL}_2(q)$, $q = 11, 13, 19, 29, 37$, $\mathrm{PSL}_3(4)$, $\mathrm{PSU}_3(3)$
$p = 5$	M_{22}, Alt_7, $\mathrm{PSL}_2(q)$, $q = 9, 19, 25, 29$, $\mathrm{PSL}_3(4)$, ${}^2B_2(8)$
$p = 7$	$\mathrm{PSL}_2(q)$, $q = 7, 13, 27, 29$, $\mathrm{PSL}_3(4)$, $\mathrm{PSU}_3(3)$
$p = 13$	$\mathrm{PSL}_2(27)$
$p = 19$	$\mathrm{PSL}_2(37)$

(The case $\mathrm{PSL}_2(19)$ for characteristic 19 was proved by Burness and Testerman. Several that act irreducibly on $L(G)$ can be deduced from $\mathrm{Hom}(\Lambda^2(L(G)), L(G))$ being 1-dimensional on restriction to H.)

So what Lie-primitive subgroups are known to exist?

We just give one example here: $G = F_4$.

p	H
$p \nmid \lvert H\rvert$	$\mathrm{PSL}_2(q)$, $q = 8, 13, 17, 25, 27$, $\mathrm{PSL}_3(3)$, ${}^3D_4(2)$
$p = 2$	$\mathrm{PSL}_2(q)$, $q = 25, 27$, $\mathrm{PSL}_3(3)$, $\mathrm{PSL}_4(3)$
$p = 3$	$\mathrm{PSL}_2(q)$, $q = 13, 25$, ${}^3D_4(2)$
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