Reductive Subgroups of Reductive Groups

Alastair Litterick,
University of Bielefeld

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Subgroup Structure of Simple Algebraic Groups

Set-up

- K: algebraically closed field, characteristic $p \geq 0$.
- G: Simple algebraic group over K.

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- 1990s onwards:
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  - Various extensions beyond maximal subgroups.
  - Serre introduces G-complete reducibility.
- Open problem: Understand all reductive subgroups.

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If $G = GL_n(K)$ then this is true representation theory of H.

Complete Reducibility

Definition

A subgroup $H \leq P$ is *G-completely reducible* if:

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For $G = GL(V)$:
Parabolic subgroup $=$ Stabiliser of a flag

$$0 = V_0 \leq V_1 \leq \dots V_r = V$$

$$P = \left(\begin{array}{c|c|c|c} A_1 & * & * & * \\ \hline \cdot & A_2 & * & * \\ \hline \cdot & \cdot & \ddots & * \\ \hline \cdot & \cdot & \cdot & A_r \end{array}\right)$$

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For $G = GL(V)$:
Levi subgroup = Stabiliser of a flag and a 'complement'

$$V = V'_0 \geq \ldots \geq V'_r = 0 \quad \text{where} \quad V = V_i \oplus V'_i \;\; \forall\, i.$$

$$P = \left(\begin{array}{c|c|c|c} A_1 & * & * & * \\ \hline \cdot & A_2 & * & * \\ \hline \cdot & \cdot & \ddots & * \\ \hline \cdot & \cdot & \cdot & A_r \end{array}\right), \qquad L = \left(\begin{array}{c|c|c|c} A_1 & \cdot & \cdot & \cdot \\ \hline \cdot & A_2 & \cdot & \cdot \\ \hline \cdot & \cdot & \ddots & \cdot \\ \hline \cdot & \cdot & \cdot & A_r \end{array}\right)$$

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X is G-completely reducible

$\Leftrightarrow$

X is L-irreducible for some Levi subgroup L of G

Reductive Subgroups

Two problems:

- Classify L-irreducible subgroups for each Levi L of G.
- For each parabolic $P = Q \rtimes L$, and each L-irreducible reductive subgroup X, understand *complements* to Q in QX:

$$QX_1 = QX, \qquad Q \cap X_1 = 1$$

Complements and Cohomology

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Complements to Q in such a semidirect product have the form

$$\{(\phi(x), x) \,:\, x \in X\}$$

where $\phi : X \to Q$ is a *1-cocycle*:

$$\phi(x_1 x_2) = \phi(x_1)\,(x_1 \cdot \phi(x_2))$$

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Define $Z^1(X, Q) = \{\text{cocycles } X \to Q\}$.

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$$Z^1(X, Q) = \{\text{cocycles } X \to Q\}$$

Complements corresponding to ϕ_1 and ϕ_2 are conjugate in QX iff there exists $q \in Q$ with

$$\phi_1(x) = q^{-1}\phi_2(x)(x \cdot q) \quad \forall\, x \in X$$

equivalence classes are *first cohomology classes*, $H^1(X, Q)$.

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- If Q is abelian then $Z^1(X, Q)$ and $H^1(X, Q)$ are vector spaces.

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Example

X : Simple algebraic group. V: X-module.

$$G = GL(V \oplus K), \quad P = \mathrm{Stab}_G(V), \quad L = \mathrm{Stab}_G(V \oplus K).$$

Take $X < L$. Then $Q \cong V$ as X-modules,
hence $H^1(X, Q)$ is a vector space.

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Then $X < L$, and Q is nilpotent of class 2:

$$Z = Z(Q) = [Q, Q] \cong V, \quad Q/[Q, Q] \cong V \oplus K$$

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$$Z = Z(Q) = [Q,Q] \cong V, \quad Q/[Q,Q] \cong V \oplus K$$

and $H^1(X,Q)$ fits into an exact sequence:

$$0 \to H^0(X,Z) \to H^0(X,Q) \to H^0(X,Q/Z)$$
$$\to H^1(X,Z) \to H^1(X,Q) \to H^1(X,Q/Z) \to H^2(X,Z)$$

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It turns out that $H^1(X, Q)$ 'looks like' a pair of lines!

Theorem (Liebeck, Seitz)

If G is simple of exceptional type and $p > 7$, then all reductive subgroups are G-cr.

Results

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Theorem (L., Thomas)

If G is simple of exceptional type and p is good for G then non-G-cr reductive subgroups are classified.

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Observation

In every case, $H^1(X, Q)$ 'looks like' an affine variety.

Conjecture

Let X be reductive, and let Q be a unipotent affine algebraic X-group, with a central filtration

$$Q = Q(1) \geq Q(2) \geq \ldots \geq Q(r+1) = 0$$

where each section $Q(i)/Q(i+1) =: V_i$ is a rational X-module.

Then $H^1(X, Q)$ is a finite union of subspaces of

$$\bigoplus_{i=1}^{r} H^1(X, V_i).$$

Moreover, the maps in the long exact sequence of cohomology are morphisms of affine varieties.

Bad characteristic example

Example

$$G = E_8, \quad p = 2, \quad P = Q \rtimes L, \quad L = E_6 T_2.$$

E_6 has a well-known irreducible subgroup $X = D_4$.

Q is nilpotent of class 5:

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$$Q = Q(1) \rhd Q(2) \rhd Q(3) \rhd Q(4) \rhd Q(5) \rhd Q(6) = 1,$$

$$\begin{aligned} Q(1)/Q(2) &\cong V \oplus K \oplus K, \\ Q(2)/Q(3) &\cong V \oplus K, \\ Q(3)/Q(4) &\cong V \oplus K, \\ Q(4)/Q(5) &\cong K, \\ Q(5)/Q(6) &\cong K \end{aligned}$$

where V is irreducible of dimension 26 and $H^1(X, V) = K^2$.

The Conjecture is Probably False but Also Sort-of True

Known:

- $Z^1(X, Q)$ is a closed subset of Q^N for some $N > 0$.
- $H^1(X, Q)$ is the quotient of $Z^1(X, Q)$ under an action of Q.
- There is a sub-torus of G acting on $H^1(X, Q)$.

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But this isn't enough!

Example

Let $(K, +)$ act on K^2 via $\lambda \cdot (x, y) = (x, y + \lambda x)$.

Orbits are singletons $\{(0, y)\}$ and vertical lines $x = a \neq 0$.

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Ongoing: How much can we rescue?
(Étale slices, other types of quotients, larger categories, . . .)

Reductive subgroups: Their Classification So Far

$G =$	E_8	E_7	E_6	F_4	G_2
$X = A_1$	2 3 5 7	2 3 5 7	2 3 5	2 3	2
A_2	2 3	2 3	2 3	3	
B_2	2 5	2	2	2	
G_2	2 3 7	2 7			
A_3	2	2			
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C_3	3				
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D_4	2	2			

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Red: Understood (Liebeck, Saxl, Testerman)

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