Complexity questions for classes of closed subgroups of $\mathrm{Sym}(\mathbb{N})$

André Nies

THE UNIVERSITY OF AUCKLAND
NEW ZEALAND

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Borelness of classes of closed subgroups

$\mathrm{Sym}(\mathbb{N})$ is the topological group of permutations of $\mathbb{N}$.
We consider classes $\mathcal{C}$ of closed subgroups of $\mathrm{Sym}(\mathbb{N})$:

- compact (i.e., profinite),
- locally compact,
- oligomorphic (for each n only finitely many n-orbits)
- topologically finitely generated, ...

First we ask whether the class is Borel. This means that the groups in the class form points in a space that can be investigated using descriptive set theory.

Leading questions

Suppose a class $\mathcal{C}$ of closed subgroups of $\mathrm{Sym}(\mathbb{N})$ is Borel.
Given $G, H \in \mathcal{C}$.

- How complicated is it to recognise whether G, H are conjugate?
- How complicated is it to recognise whether G, H are (topologically) isomorphic?

What do you mean by "how complicated"?
One can compare them to benchmark equivalence relations:

- $\mathtt{GI}$ isomorphism of countable graphs.
- E_0, almost equality of infinite bit sequences
- $\mathtt{id}_{\mathbb{R}}$, identity of reals

Borel reducibility $\leq_B$

- Let X, Y be “standard Borel spaces” (X, Y carry Borel structures of uncountable Polish spaces). A function $g\colon X \to Y$ is Borel if the preimage of each Borel set in Y is Borel in X.
- Let E, F equivalence relations on X, Y respectively. We write $E \leq_B F$ (Borel below) if there is a Borel function $g\colon X \to Y$ such that

$$uEv \leftrightarrow g(u)Fg(v)$$

  for each $u, v \in X$.[1]
- Write $E \equiv_B F$ (Borel equivalent) if $E \leq_B F \leq_B E$.

For instance, $\mathtt{id}_{\mathbb{R}} \equiv_B \mathtt{id}_Y$ for an arbitrary uncountable Polish space Y. We have $\mathtt{id}_{\mathbb{R}} <_B E_0 <_B \mathtt{GI}$.

[1] See S. Gao, Invariant descriptive set theory, 2009

The space of closed subgroups of $\mathrm{Sym}(\mathbb{N})$

The closed subgroups of $\mathrm{Sym}(\mathbb{N})$ can be seen as points in a standard Borel space. To define the Borel sets, we start with sets of the form

$$\{G \leq_c \mathrm{Sym}(\mathbb{N})\colon G \cap N_\sigma \neq \emptyset\},$$

where

- σ is a 1-1 map $\{0, \ldots, n-1\} \to \mathbb{N}$
- $N_\sigma = \{\alpha \in \mathrm{Sym}(\mathbb{N})\colon \sigma \prec \alpha\}$

The Borel sets are generated from these basic sets by complementation and countable union.

For instance, for every $\alpha \in \mathrm{Sym}(\mathbb{N})$ we have the Borel set $\bigcap_k \{H\colon H \cap N_{\alpha\restriction_k} \neq \emptyset\}$ which says that the closed subgroup contains α.

Borelness of classes of groups

Recall we consider classes $\mathcal{C}$ of closed subgroups G of $\mathrm{Sym}(\mathbb{N})$:

(a) compact (i.e., profinite)

(b) locally compact

(c) oligomorphic (for each n only finitely many n-orbits)

(d) topologically finitely generated

(e) ....

The first three classes are known to be Borel.
E.g. for (a) and (b), given G consider the tree $\{\sigma : G \cap N_\sigma \neq \emptyset\}$.

The class (d) is not known to be Borel. Within the profinite groups, being f.g. is Borel.

Topologically finitely generated profinite groups I

Theorem

The isomorphism relation $E_{f.g.}$ between finitely generated profinite groups is Borel-equivalent to $\mathtt{id}_{\mathbb{R}}$.

$\mathtt{id}_{\mathbb{R}} \leq_B E_{f.g.}$: Let $\widehat{\mathbb{Z}}$ be the profinite completion of the ring $\mathbb{Z}$. For any set P of primes, let[2]

$$G_P = \prod_{p \in P} \mathtt{SL}_2(\mathbb{Z}_p) = \mathtt{SL}_2(\widehat{\mathbb{Z}}) / \prod_{q \notin P} \mathtt{SL}_2(\mathbb{Z}_q).$$

$P = Q \leftrightarrow G_P \cong G_Q$.
$P \to G_P$ is a Borel map.

[2] Lubotzky (2005), Prop 6.1

Topologically finitely generated profinite groups II

The isomorphism relation $E_{f.g.}$ between finitely generated profinite groups is Borel equivalent to $\mathtt{id}_{\mathbb{R}}$.

$E_{f.g.} \leq_B \mathtt{id}_{\mathbb{R}}$ (smoothness):

- A finitely generated profinite group G is determined by its isomorphism types of finite quotients.
- Let $q(G)$ be the set of these isomorphism types, written in some fixed way as an infinite bit sequence. This map is Borel because from G one can "determine" its finite quotients[3].
- Then $G \cong H \iff q(G) = q(H)$. So $E_{f.g.}$ is smooth.

Isomorphism of residually finite f.g. groups is complicated ("weakly universal", Jay Williams '15) and hence not smooth. So taking profinite completion loses information (new proof of a known fact).

[3] e.g. Fried/Jarden, Field arithmetic, 16.10.7

Graph isomorphism $\leq_B$ isomorphism of profinite groups

A group G is nilpotent-2 if it satisfies the law $[[x, y], z] = 1$.

Let $\mathcal{N}_2^p$ denote the variety of nilpotent-2 groups of exponent p.

Theorem

Let $p \geq 3$ be prime. Graph isomorphism can be Borel reduced to isomorphism between profinite $\mathcal{N}_2^p$ groups.

Proof: A result of Alan Mekler (1981) implies the theorem for countable abstract groups. We adapt his construction to the profinite setting.

A symmetric and irreflexive countable graph is called nice if it has no triangles, no squares, and for each pair of distinct vertices x, y, there is a vertex z joined to x and not to y.

Mekler's construction

Nice graph isomorphism $\leq_B$ isomorphism of countable groups in $\mathcal{N}_2^p$.

- Let F be the free $\mathcal{N}_2^p$ group on free generators $x_0, x_1, \ldots$.
- For $r \neq s$ we write $x_{r,s} = [x_r, x_s]$.
- Given a graph with domain $\mathbb{N}$ and edge relation A, let

$$G(A) = F / \langle x_{r,s} \colon \; rAs \rangle_{\text{normal closure}}.$$

- The centre of $G(A)$ is abelian of exponent p with a basis consisting of the $x_{r,s}$ such that $\neg rAs$.

Show that A can be reconstructed from $G(A)$. Therefore:

Let A, B be a nice graphs. Then $A \cong B$ iff $G(A) \cong G(B)$.

Profinite version of Mekler's construction

- Elements of $G(A) = F/\langle x_{r,s} \colon rAs \rangle$ have unique normal form
$$\prod_{\langle r,s\rangle \in L} x_{r,s}^{\beta_{rs}} \prod_{i\in D} x_i^{\alpha_i},\ 0 < \alpha_i, \beta_{rs} < p,$$
where L is a finite set of non-edges, D a finite set of vertices.
- Let R_n be the normal subgroup of $G(A)$ generated by the x_i, $i \geq n$. Let $\overline{G}(A)$ be the completion of $G(A)$ w.r.t. the R_n, i.e.,
$$\overline{G}(A) = \varprojlim_n G(A)/R_n.$$
- Each $G(A)/R_n$ is finite, so this is a profinite group.
- Elements have normal form $\prod_{\langle r,s\rangle \in L} x_{r,s}^{\beta_{rs}} \prod_{i\in D} x_i^{\alpha_i}$, where L and D are now allowed to be infinite.

Verify that A can be reconstructed from $\overline{G}(A)$:

Let A, B be a nice graphs. Then $A \cong B$ iff $\overline{G}(A) \cong \overline{G}(B)$.

$A \to \overline{G}(A)$ is Borel. So $\texttt{GI} \leq_B$ isomorphism of profinite $\mathcal{N}_2^p$ groups.

A condition implying that isomorphism on $\mathcal{C}$ is Borel below graph isomorphism

Lemma (with Kechris and Tent)

Let $\mathcal{C}$ be Borel class of closed subgroups, with $\mathcal{C}$ closed under conjugation in $\mathrm{Sym}(\mathbb{N})$.

- For $G \in \mathcal{C}$ suppose $\mathcal{N}_G$ is a countably infinite set of open subgroups of G that forms a nbhd basis of 1.
- Suppose the relation $\{\langle G, U\rangle \colon U \in \mathcal{N}_G\}$ is Borel, and isomorphism invariant in the sense that $\phi \colon G \cong H$ implies $U \in \mathcal{N}_G \iff \phi(U) \in \mathcal{N}_H$.

Then isomorphism on $\mathcal{C}$ is Borel reducible to graph isomorphism.

We will apply this to

(1) $\mathcal{C}$ = locally compact; $\mathcal{N}_G$= compact open subgroups of G

Proof of Lemma

Lemma (recall). For $G \in \mathcal{C}$ suppose $\mathcal{N}_G$ is a countably infinite set of open subgroups of G that forms a nbhd basis of 1.
Suppose the relation $\{\langle G, U\rangle \colon U \in \mathcal{N}_G\}$ is Borel, and isomorphism invariant in the sense that $\phi\colon G \cong H$ implies $U \in \mathcal{N}_G \iff \phi(U) \in \mathcal{N}_H$.
Then isomorphism on $\mathcal{C}$ is Borel reducible to graph isomorphism.

- To $G \in \mathcal{C}$ we can Borel assign a list $C_0, C_1, \ldots$ of the cosets of all the $U \in \mathcal{N}_G$ (using Borelness of the relation "$U \in \mathcal{N}_G$").
- G acts on the cosets from the left. For $g \in G$ let $\widehat{g} \in \mathrm{Sym}(\mathbb{N})$ be the corresponding permutation of indices of cosets.
- $g \mapsto \widehat{g}$ is a topological embedding $G \cong \widehat{G} \leq_c \mathrm{Sym}(\mathbb{N})$. The map $G \mapsto \widehat{G}$ is Borel.
- $G \cong H \iff \widehat{G}$ conjugate to $\widehat{H}$.
- Fact from descriptive set theory: every orbit eqrel of a Borel $\mathrm{Sym}(\mathbb{N})$ action is $\leq_B$ graph isom.

Theorem (with Kechris and Tent)

- Isomorphism of t.d.l.c. groups is Borel equivalent to graph isomorphism. (Asked by P.E. Caprace.)
- Same for conjugacy.
- Isomorphism of oligomorphic groups is Borel below graph isomorphism.

$G \leq_c \mathrm{Sym}(\mathbb{N})$ is Roelcke precompact if for each open subgroup U there is finite $F \subseteq G$ such that $UFU = G$.

- Same as inverse limit of an ω-chain of oligomorphic (on some countable set) groups (Tsankov)
- Roelcke precompact $\Rightarrow$ countably many open subgroups. So isomorphism of Roelcke precompact is $\equiv_B$ graph isomorphism.

Independently Rosendal and Zielinski (on arXiv Oct 2016).

Oligomorphic groups

The conjugacy relation for oligomorphic groups is smooth.

To see this,

- given G let M_G be the corresponding orbit equivalence structure: introduce a $2n$-ary relation for each $n > 0$, which holds for two n-tuples of distinct elements if they are in the same orbit.
- M_G is ω-categorical.
- G, H are conjugate $\iff M_G \cong M_H$.
- Isomorphism of ω-categorical structures is smooth: take as a real invariant the first-order theory.

NB: The previous result doesn't show $\cong$ on oligormorphic is smooth: $\widehat{G}$ obtained there is not oligomorphic (even though $G \cong \widehat{G}$).

Questions

- How complex is isomorphism of arbitrary closed subgroups of S_∞? Is it $\leq_B$-complete for analytic equivalence relations?
- Characterise $G \leq_c \mathrm{Sym}(\mathbb{N})$ with only countably many open subgroups. (E.g. $\mathtt{PSL}_2(\mathcal{Q}_p)$ is another example by a result of Tits.)
- How complex is isomorphism of oligomorphic groups?
  Evans and Hewitt: every profinite group is a topological quotient of an oligomorphic group. This may indicate it's complicated.
- How about if the language of the corresponding ω-categorical orbit structure can be made finite? (For a finite language, isomorphism is a Borel equivalence relation with all classes countable.)

Reference for the results on profinite groups:

N., Complexity of isomorphism between profinite groups, arXiv: