Flat groups and graphs

The unreasonable connectedness of mathematics

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Overview

Locally Compact Groups

Suppose G is a locally compact group (lcg).

The connected component containing the identity, G_0, is a normal subgroup of G.

$$G_0 \hookrightarrow G \twoheadrightarrow G/G_0$$

Both G_0 and G/G_0 are locally compact.

The quotient G/G_0 is a totally disconnected group.

Every locally compact group is connected by totally disconnected.

Connected and Totally Disconnected LCGs

Connected lcgs are studied via *approximation by Lie groups*.

Theorem (Gleason, 1952)

Let G be a connected locally compact group and $\mathcal{N}$ be a neighbourhood of the identity.
Then G has a compact, normal subgroup $K \subset \mathcal{N}$ such that G/K is a real Lie group.

Totally disconnected locally compact groups not so nice.

Theorem (van Dantzig, 1936)

Let G be a totally disconnected locally compact group and $\mathcal{N}$ be a neighbourhood of the identity.

Then G has a compact, open subgroup $\mathcal{O} \subset \mathcal{N}$.

(If we knew $\mathcal{O} \triangleleft G$ then could approximate G by discrete groups.)

A deep understanding of arbitrary tdlc groups is the missing piece in the puzzle of locally compact groups.

Totally Disconnected Locally Compact Groups

G.A. Willis, Math. Ann. 300 (1994) 341–363

The structure of totally disconnected locally compact groups.

Compact open subgroups play a key role in the structure theory.

Key idea:

We use linear algebra to describe and analyze connected locally compact groups (via Lie algebras of approximating Lie groups).

Eigenvalues and eigenspaces are powerful tools in the study of linear operators.

Develop analogues for automorphisms of tdlc groups.

The Scale of an Automorphism

Suppose G is a tdlc group and $\alpha \in \mathrm{Aut}(G)$. The scale of α is

$$s(\alpha) = \min_{V \text{cpt open} \leq G} |\alpha(V) : \alpha(V) \cap V| .$$

Lemma (Willis)

$$s(\alpha) \in \mathbb{N}$$

Proof.

Since $s(\alpha)$ is an index, $s(\alpha) \in \mathbb{N} \cup \infty$. Need to show $s(\alpha) < \infty$.
Since V is compact and open, $\alpha(V)$ is compact and open.
Hence $\alpha(V) \cap V$ is open, and so are its cosets in $\alpha(V)$.
The cosets form an open cover of $\alpha(V)$.
Compactness of $\alpha(V)$ implies the existence of a finite subcover, so the index $|\alpha(V) : \alpha(V) \cap V|$ must be finite.

□

Tidy Subgroups

A compact open subgroup U is tidy for $\alpha \in \mathrm{Aut}(G)$ if

$$s(\alpha) = |\alpha(U) : \alpha(U) \cap U| .$$

Starting with $\alpha \in \mathrm{Aut}(G)$ and an arbitrary compact open subgroup V, there is a procedure (due to Willis) guaranteed to produce a subgroup tidy for α in a finite number of steps.

Theorem (Willis)

U is tidy for $\alpha \Leftrightarrow$ U is tidy for α^{-1}.

Idea: think of α as a bounded linear operator,
$s(\alpha)$ as a spectral radius or eigenvalue,
and U as an eigenspace.

We'll be wrong, but we can exploit the analogy.

The Scale Function on the Group

By considering inner automorphisms, the scale function induces a map $s\colon G \to \mathbb{N}$.

We say U is tidy for $x \in G$ if U is tidy for conjugation by x.

G is uniscalar if $s(x) = 1$ for all $x \in G$.
Every compact group is uniscalar since the group itself is invariant under conjugation, hence is tidy for every element.

Theorem (Willis)

The map $s\colon G \to \mathbb{N}$ is continuous and satisfies

- *$s(h^n) = s(h)^n$ for all $n \in \mathbb{N}$ and $h \in G$, and*
- *$\Delta(h) = s(h)/s(h^{-1})$ where Δ is the modular function on G.*

Example

Suppose $\mathcal{T}_q$ is a homogeneous tree of valency $q+1$ and $G = \text{Aut}(\mathcal{T}_q)$. G is unimodular, but not uniscalar.

- If $y \in G$ is an automorphism of $\mathcal{T}_q$ that fixes a vertex v on the tree, then
$$s(y) = 1$$
and $stab_G(v)$ is tidy for y.
- If $x \in G$ is a hyperbolic automorphism that shifts by one edge in a given direction, then
$$s(x) = q, \quad s(x^{-1}) = q$$
and the fixator of any string of edges on the axis of translation of x is tidy for both x and x^{-1}.

When the analogy fails (Willis)

Consider $\alpha \in \text{Aut}(G)$ with

$$s(\alpha) = \min_{V \text{cpt open} \leq G} |\alpha(V) \colon \alpha(V) \cap V|\,.$$

In general, $\quad s(\alpha^{-1}) \neq s(\alpha)^{-1}. \quad$ Indeed,

- $s(\alpha^{-1}) = s(\alpha)^{-1} \Rightarrow s(\alpha) = s(\alpha^{-1}) = 1$
- $s(\alpha) = 1$ if and only if $\alpha(V) \subseteq V$ for some compact open subgroup of G.
- $s(\alpha) = 1 = s(\alpha^{-1})$ if and only if $\alpha(V) = V$ for some compact open subgroup of G.
- given $(m, n) \in \mathbb{N} \times \mathbb{N}$, can construct α s.t. $s(\alpha) = m$ and $s(\alpha^{-1}) = n$.

Möller's characterisation of tidiness

Suppose G is a tdlc group.
Fix $x \in G$ and a compact open $U \leq G$.

Let $\Omega = U \backslash G$ be the space of right cosets of U in G.
Denote by $\nu = U \in \Omega$ the trivial coset.
Construct a graph Γ_+ by

$$V\Gamma_+ = \bigcup_{i \geq 0} \nu x^i U \quad \text{and} \quad E\Gamma_+ = \bigcup_{i \geq 0} (\nu x^i, \nu x^{i+1})U,$$

where $(\nu x^i, \nu x^{i+1})U = \big\{(\nu x^i u, \nu x^{i+1} u) \mid u \in U\big\}$.

Theorem (Möller, 2002)

U is tidy for $x \Leftrightarrow \Gamma_+$ is a directed regular rooted tree with all edges directed away from ν.

Möller's characterisation of tidiness

If U is tidy for x then

- U is tidy for x^n, with $s(x^n) = s(x)^n$ for $n \in \mathbb{N}$, and
- U is tidy for x^{-1}.

Möller's theorem provides a characterisation of tidiness of U for

- the subgroup $\langle x \rangle \leq G$, and
- the semigroup $\langle x \rangle_+ \subseteq G$.

Note that

- U is tidy for every element in $\langle x \rangle_+$ and
- the scale is multiplicative on $\langle x \rangle_+$.

Common tidy subgroups—Flat subgroups

A subgroup $H \leq \mathrm{Aut}(G)$ is flat if there is a compact open subgroup U that is tidy for every $\alpha \in H$.

(Think of this as simultaneous triangularisation of matrices.)

The set of uniscalar elements of H,

$$H_1 = \left\{\alpha \in H \mid s(\alpha) = 1 = s(\alpha^{-1})\right\},$$

is a subgroup of H because $\alpha \in H_1$ if and only if $\alpha(U) = U$.

Theorem (Willis)

Let H be a finitely generated flat subgroup of G. Then $H_1 \lhd H$ and $H/H_1 \cong \mathbb{Z}^r$ for some $r \in \mathbb{N}$, called the flat rank of H.

Example

Let $G = \mathbb{Q}_p^k \rtimes \mathbb{Z}^k$, where the action of $\mathbb{Z}^k$ on $\mathbb{Q}_p^k$ is

$$(n_1, \ldots, n_k).(y_1, \ldots, y_k) = (p^{-n_1}y_1, \ldots, p^{-n_k}y_k)$$

for $n_j \in \mathbb{Z}^k$ and $y_j \in \mathbb{Q}_p$. Then

- $H = (1, \mathbb{Z}^k) \cong \mathbb{Z}^k$ is a flat subgroup of G and
- $U = (\mathbb{Z}_p^k, 1) \cong \mathbb{Z}_p^k$ is tidy for H.

For $x = (n_1, \ldots, n_k) \in H$ put $m(x) = \sum_{n_j \geq 0} n_j$.

Then the scale of x is

$$s(x) = p^{m(x)}.$$

When will the scale be multiplicative?

Let G be a tdlc group with scale function $s\colon G \to \mathbb{N}$.
A semigroup $P \subseteq G$ is scale-multiplicative, or s-multiplicative, if

$$s(xy) = s(x)s(y) \text{ for every } x, y \in P.$$

Suppose x and x^{-1} both belong to P, then

$$s(x)s(x^{-1}) = s(xx^{-1}) = s(e_G) = 1.$$

Since $s(x), s(x^{-1}) \in \mathbb{N}$ this means $s(x) = s(x^{-1}) = 1$.

Working modulo the uniscalar group of a tdlcg,
the natural extension of Möller's result is to
s-multiplicative semigroups P satisfying $P \cap P^{-1} = \{e_G\}$.

Extending Möller — Praeger, Ramagge, and Willis

We build an object Γ_P and prove the following theorem.

Theorem (Praeger, Ramagge and Willis)

Suppose G is a tdlc group and $H \cong \mathbb{Z}^r$ is a flat subgroup of G.

Let P be a maximal s-multiplicative subsemigroup of H satisfying $P \cap P^{-1} = \{e_G\}$ and U be a compact open subgroup of G. Then

U is tidy for $P \Rightarrow \Gamma_P$ is a regular, rooted, strongly-simple P-graph.

What is a P-graph? | What do the adjectives mean? | How do you build Γ_P? | Example

What is a P-graph? (Brownlowe-Sims-Vitadello)

Suppose H is a finitely-generated group and P is a subsemigroup of H with $P \cap P^{-1}$ trivial.

A P-graph (Λ, d) is

- a countable category Λ, in particular $\Lambda = \mathrm{Hom}(\Lambda)$, $\mathrm{Obj}(\Lambda) \subseteq \Lambda$, and $\mathrm{dom}, \mathrm{cod}\colon \Lambda \to \mathrm{Obj}(\Lambda)$
- together with a functor $d : \Lambda \to P$, called the *degree*, which satisfies the factorization property: for every $\lambda \in \Lambda$ and $x, y \in P$ with $d(\lambda) = xy$ there are unique elements $\lambda_1, \lambda_2 \in \Lambda$ such that $\lambda = \lambda_1\lambda_2$ and $d(\lambda_1) = x,\ d(\lambda_2) = y$.

Examples

- A directed graph is a P-graph with $P = \mathbb{N}$.
- A k-graph, in the sense of Kumjian-Pask, is an $\mathbb{N}^k$-graph.

Theorem

What do the adjectives mean?

Let Λ be a P-graph.

For each $\alpha \in \mathrm{Obj}(\Lambda)$ the descendant P-graph Λ^α has

$$\begin{aligned} \mathrm{Obj}(\Lambda^\alpha) &= \{\beta \in \mathrm{Obj}(\Lambda) \mid \exists\lambda \in \mathrm{Hom}(\Lambda) \text{ with } \lambda : \alpha \to \beta\} \\ \mathrm{Hom}(\Lambda^\alpha) &= \{\lambda \in \mathrm{Hom}(\Lambda) \mid \mathrm{dom}(\lambda), \mathrm{cod}(\lambda) \in \mathrm{Obj}(\Lambda^\alpha)\}\,. \end{aligned}$$

An object α with $\Lambda^\alpha = \Lambda$ is a generator for Λ.
If α is unique it is the root of Λ and we say Λ is rooted.

Λ is strongly simple if there is at most one morphism $\lambda : \alpha \to \beta$ for any $\alpha, \beta \in \mathrm{Obj}(\Lambda)$.

Λ is regular if for every $\alpha, \beta \in \mathrm{Obj}(\Lambda)$ there is an isomorphism $\phi : \Lambda^\alpha \to \Lambda^\beta$.

Theorem

How do we build Γ_P?

Lemma
Let $H \cong \mathbb{Z}^r$ be a flat subgroup of a tdlc group G. Then there exist maximal subsemigroups P of H satisfying $P \cap P^{-1} = \{e_G\}$ and such P are finitely generated.

Fix P; let $\Sigma = \{x_1, \ldots, x_n\}$ be smallest generating set for P.

Let Ω be the coset space $U\backslash G$. In Ω, denote U by ν.
For each $x \in P$, the U-orbit $\nu x U$ is a finite subset of Ω. Let

$$\begin{aligned} V(\Gamma_P) &= \bigcup_{x \in P} \nu x U \\ E(\Gamma_P) &= \bigcup_{i \in \{1,\ldots,n\}} \{(\nu x, \nu x x_i)U \mid x \in P,\ x_i \in \Sigma\}\,. \end{aligned}$$

Construct Γ_P as a "path space" from these ingredients.

Theorem

Interesting example of Γ_P

Let $G = \mathbb{Q}_p^2 \rtimes \mathbb{Z}^2$ where the action of $\mathbb{Z}^2$ on $\mathbb{Q}_p^2$ is defined by extending the following actions of the standard basis vectors:

$$(1,0)\cdot(a,b) = (p^{-1}a, p^{-1}b) \text{ and } (0,1)\cdot(a,b) = (p^{-1}a, pb)$$

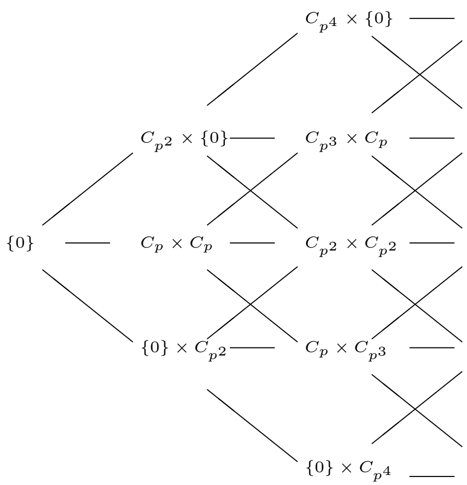
for $a, b \in \mathbb{Q}_p$. Then $H = (1, \mathbb{Z}^2) \cong \{(n_1, n_2) \mid n_j \in \mathbb{Z}\}$ is flat, and $U = (\mathbb{Z}_p^2, 1)$ tidy for H.

The maximal s-multiplicative subsemigroups of H are all isomorphic and are not isomorphic to $\mathbb{N}^2$ because their minimal generating sets have three elements.

The P-graph Γ_P is not a product of trees.

Theorem

Interesting example of Γ_P



Theorem

Any Questions?

Thank you for your attention.