Reduced models for linearly elastic thin films allowing for fracture, debonding or delamination

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Presentation of the 3D model

The system

$$\Omega^\varepsilon = \Omega^\varepsilon_f \cup \Omega^\varepsilon_b \cup \Omega^\varepsilon_s$$

is made of

- a film $\Omega^\varepsilon_f := \omega \times (0,\varepsilon)$ ($\omega \subset \mathbb{R}^2$ open, bounded) deposited on
- an infinite substrate $\Omega^\varepsilon_s := \omega \times (-\infty,-\varepsilon)$ through
- a bonding layer $\Omega^\varepsilon_b := \omega \times [-\varepsilon,0]$.

✓ Kinematics : isotropic linearized elasticity (displacement $v : \Omega^\varepsilon \to \mathbb{R}^3$) ;
✓ Loadings : inelastic strain e_0, displacement in the substrate v_0 ;
✓ Cracks : sets $\Gamma \subset \overline{\Omega}^\varepsilon$ of finite area ;
✓ Total energy : for all $\Gamma \subset \overline{\Omega}^\varepsilon$ and $v : \Omega^\varepsilon \setminus \Gamma \to \mathbb{R}^3$ with $v = v_0$ in Ω^ε_s,

$$J(\varepsilon)(v,\Gamma) = \int_{\Omega^\varepsilon\setminus\Gamma} \mathbb{C}^\varepsilon(e(v)-e_0) : (e(v)-e_0)\, dx + \int_\Gamma \kappa^\varepsilon \, d\mathcal{H}^2,$$

where $e(v) = (\nabla v + \nabla v^T)/2$ is the linearized strain.

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$$J(\varepsilon)(v) = \int_{\Omega^\varepsilon \setminus J_v} \mathbb{C}^\varepsilon e(v) : e(v)\, dx + \int_{J_v} \kappa^\varepsilon \, d\mathcal{H}^2,$$

where $Ev = (Dv + Dv^T)/2 = e(v)dx + (v^+ - v^-) \odot \nu_v \delta_{J_v}$.
We assume that

$$\mathbb{C}^\varepsilon = \underbrace{\mathbb{C}_f}_{(\lambda_f,\mu_f)} \chi_{\Omega^\varepsilon_f} + \varepsilon^2 \underbrace{\mathbb{C}_b}_{(\lambda_b,\mu_b)} \chi_{\Omega^\varepsilon_b}, \quad \kappa^\varepsilon = \kappa_f \chi_{\Omega^\varepsilon_f} + \varepsilon \kappa_b \chi_{\Omega^\varepsilon_b}.$$

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$$\begin{aligned} J(\varepsilon)(v) &= \int_{\Omega^\varepsilon_f \setminus J_v} \mathbb{C}_f e(v) : e(v)\, dx + \kappa_f \mathcal{H}^2(J_v \cap \Omega^\varepsilon_f) \\ &\quad + \varepsilon^2 \int_{\Omega^\varepsilon_b \setminus J_v} \mathbb{C}_b e(v) : e(v)\, dx + \varepsilon \kappa_b \mathcal{H}^2(J_v \cap \Omega^\varepsilon_b), \end{aligned}$$

where $Ev = (Dv + Dv^T)/2 = e(v)dx + (v^+ - v^-) \odot \nu_v \delta_{J_v}$.

Scaling

$$\Omega = \Omega^1,\ \Omega_f := \Omega_f^1,\ \Omega_b := \Omega_b^1,\ \Omega_s := \Omega_s^1$$

For all $x = (x', x_3) \in \Omega$,

$$u_\alpha(x', x_3) = v_\alpha(x', \varepsilon x_3), \quad u_3(x', x_3) = \varepsilon v_3(x', \varepsilon x_3).$$

Then for all $u \in SBD(\Omega)$ with $v = 0$ in Ω_s,

$$J_\varepsilon(u) = \varepsilon^{-1} J(\varepsilon)(v) = J_\varepsilon^f(u) + J_\varepsilon^b(u),$$

where

$$J_\varepsilon^f(u) := \int_{\Omega_f \setminus J_u} \mathbb{C}_f e^\varepsilon(u) : e^\varepsilon(u)\, dx + \kappa_f \int_{J_u \cap \Omega_f} \left| ((\nu_u)', \varepsilon^{-1}(\nu_u)_3) \right| d\mathcal{H}^2,$$

$$J_\varepsilon^b(u) := \varepsilon^2 \int_{\Omega_b \setminus J_u} \mathbb{C}_b e^\varepsilon(u) : e^\varepsilon(u)\, dx + \kappa_b \varepsilon \int_{J_u \cap \Omega_b} \left| ((\nu_u)', \varepsilon^{-1}(\nu_u)_3) \right| d\mathcal{H}^2,$$

and

$$e^\varepsilon(u) := \begin{pmatrix} e_{11}(u) & e_{12}(u) & \varepsilon^{-1} e_{13}(u) \\ e_{12}(u) & e_{22}(u) & \varepsilon^{-1} e_{23}(u) \\ \varepsilon^{-1} e_{13}(u) & \varepsilon^{-1} e_{23}(u) & \varepsilon^{-2} e_{33}(u) \end{pmatrix}.$$

Derivation of elastic foundations

Derivation of transverse cracks

Fracture, debonding and delamination

For all $u \in H^1(\Omega;\mathbb{R}^3)$ with $u = 0$ in Ω_s,

$$J^f_\varepsilon(u) := \int_{\Omega_f} \mathbb{C}_f e^\varepsilon(u) : e^\varepsilon(u)\, dx,$$

$$J^b_\varepsilon(u) := \varepsilon^2 \int_{\Omega_b} \mathbb{C}_b e^\varepsilon(u) : e^\varepsilon(u)\, dx.$$

For all $u \in H^1(\Omega;\mathbb{R}^3)$ with $u = 0$ in Ω_s,

$$
\begin{aligned}
J_\varepsilon^f(u) \quad &:= \quad \int_{\Omega_f} \left[\frac{\lambda_f}{2} e_{\alpha\alpha}(u) e_{\beta\beta}(u) + \mu_f e_{\alpha\beta}(u) e_{\alpha\beta}(u)\right] dx \\
&\quad + \varepsilon^{-2} \int_{\Omega_f} \left[\lambda_f e_{\alpha\alpha}(u) e_{33}(u) + 2\mu_f e_{\alpha 3}(u) e_{\alpha 3}(u)\right] dx \\
&\quad + \varepsilon^{-4} \int_{\Omega_f} \frac{\lambda_f + 2\mu_f}{2} e_{33}(u) e_{33}(u)\, dx, \\
J_\varepsilon^b(u) \quad &:= \quad \varepsilon^2 \int_{\Omega_b} \left[\frac{\lambda_b}{2} e_{\alpha\alpha}(u) e_{\beta\beta}(u) + \mu_b e_{\alpha\beta}(u) e_{\alpha\beta}(u)\right] dx \\
&\quad + \int_{\Omega_b} \left[\lambda_b e_{\alpha\alpha}(u) e_{33}(u) + 2\mu_b e_{\alpha 3}(u) e_{\alpha 3}(u)\right] dx \\
&\quad + \varepsilon^{-2} \int_{\Omega_b} \frac{\lambda_b + 2\mu_b}{2} e_{33}(u) e_{33}(u)\, dx.
\end{aligned}
$$

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& + \varepsilon^{-2} \int_{\Omega_f} \left[\lambda_f e_{\alpha\alpha}(u) e_{33}(u) + 2\mu_f e_{\alpha 3}(u) e_{\alpha 3}(u)\right] dx \\
& + \varepsilon^{-4} \int_{\Omega_f} \frac{\lambda_f + 2\mu_f}{2} e_{33}(u) e_{33}(u)\, dx, \\
J^b_\varepsilon(u) \quad := \quad & \varepsilon^{2} \int_{\Omega_b} \left[\frac{\lambda_b}{2} e_{\alpha\alpha}(u) e_{\beta\beta}(u) + \mu_b e_{\alpha\beta}(u) e_{\alpha\beta}(u)\right] dx \\
& + \int_{\Omega_b} \left[\lambda_b e_{\alpha\alpha}(u) e_{33}(u) + 2\mu_b e_{\alpha 3}(u) e_{\alpha 3}(u)\right] dx \\
& + \varepsilon^{-2} \int_{\Omega_b} \frac{\lambda_b + 2\mu_b}{2} e_{33}(u) e_{33}(u)\, dx \Rightarrow u_3 = 0
\end{aligned}
$$

For all $u \in H^1(\Omega; \mathbb{R}^3)$ with $u = 0$ in Ω_s,

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J^b_\varepsilon(u) \quad &:= \quad \varepsilon^{2} \int_{\Omega_b} \left[\frac{\lambda_b}{2} e_{\alpha\alpha}(u) e_{\beta\beta}(u) + \mu_b e_{\alpha\beta}(u) e_{\alpha\beta}(u)\right] dx \\
&\quad + \int_{\Omega_b} \left[\lambda_b e_{\alpha\alpha}(u) e_{33}(u) + 2\mu_b e_{\alpha 3}(u) e_{\alpha 3}(u)\right] dx \;
\begin{matrix} \nearrow \int_{-1}^{0} \partial_3 u_\alpha \, dx_3 \sim u_\alpha|_{\Omega_f} \\ \searrow \int_{-1}^{0} \partial_\alpha u_3 \, dx_3 \sim 0 \end{matrix} \\
&\quad + \varepsilon^{-2} \int_{\Omega_b} \frac{\lambda_b + 2\mu_b}{2} e_{33}(u) e_{33}(u)\, dx \Rightarrow u_3 = 0
\end{aligned}
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Theorem (B.-Henao)

J_ε "Γ-converges" to J_0, where for all $\bar{u} \in H^1(\omega;\mathbb{R}^2)$,

$$J_0(\bar{u}) = \underbrace{\int_\omega \Big[\frac{\lambda_f\mu_f}{\lambda_f+2\mu_f} e_{\alpha\alpha}(\bar{u})e_{\beta\beta}(\bar{u}) + \mu_f e_{\alpha\beta}(\bar{u})e_{\alpha\beta}(\bar{u})\Big]dx'}_{\text{usual reduced energy (Ciarlet)}} + \underbrace{\frac{\mu_b}{2}\int_\omega |\bar{u}|^2 dx'}_{\text{bonding energy}}.$$

✓ Limit displacements are not only of Kirchhoff-Love type, but actually planar (condition on the substrate) ;
✓ Recovery sequence (if $\bar{u} \in H^2(\omega;\mathbb{R}^2)$) :

$$u_\varepsilon(x',x_3) = \begin{cases} (\bar{u}(x'),0) - \varepsilon^2 x_3 \frac{\lambda_f}{\lambda_f+2\mu_f} e_{\alpha\alpha}(\bar{u})(x')\vec{e}_3 & \text{if} \quad x \in \Omega_f, \\ (x_3+1)(\bar{u}(x'),0) & \text{if} \quad x \in \Omega_b, \\ 0 & \text{if} \quad x \in \Omega_s. \end{cases}$$

For all $u \in SBD(\Omega_f)$,

$$J_\varepsilon^f(u) \quad := \quad \int_{\Omega_f} \mathbb{C}_f e^\varepsilon(u) : e^\varepsilon(u)\, dx + \kappa_f \int_{J_u \cap \Omega_f} \left| ((\nu_u)', \varepsilon^{-1}(\nu_u)_3) \right| \, d\mathcal{H}^2$$

Compactness (Bellettini-Coscia-Dal Maso) : let $(u_\varepsilon) \subset SBD(\Omega_f)$ with $\|u_\varepsilon\|_\infty \leq C$ and $J_\varepsilon^f(u_\varepsilon) \leq C$, then "$u_\varepsilon \rightharpoonup u$ in $SBD(\Omega_f)$" where

$$e_{i3}(u) = 0 \text{ in } \Omega_f, \quad (\nu_u)_3 = 0 \text{ on } J_u \cap \Omega_f \;\Rightarrow\; E_{33}u = D_3 u_3 = 0.$$

✓ Contrary to pure elasticity or plasticity u is not of Kirchhoff-Love type :

$$E_{\alpha 3} u = \frac{[u]_3 (\nu_u)_\alpha}{2} \mathcal{H}^2 \llcorner J_u \neq 0.$$

For all $u \in SBD(\Omega_f)$,

$$J^f_\varepsilon(u) := \int_{\Omega_f} \left[\frac{\lambda_f}{2} e_{\alpha\alpha}(u)e_{\beta\beta}(u) + \mu_f e_{\alpha\beta}(u)e_{\alpha\beta}(u)\right] dx$$
$$+\varepsilon^{-2} \int_{\Omega_f} \left[\lambda_f e_{\alpha\alpha}(u)e_{33}(u) + 2\mu_f e_{\alpha 3}(u)e_{\alpha 3}(u)\right] dx$$
$$+\varepsilon^{-4} \int_{\Omega_f} \frac{\lambda_f + 2\mu_f}{2} e_{33}(u)e_{33}(u)\, dx + \kappa_f \int_{J_u \cap \Omega_f} \left|((\nu_u)', \varepsilon^{-1}(\nu_u)_3)\right| \, d\mathcal{H}^2.$$

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Theorem (B.-Henao)

$$u_3 \in SBV(\omega), \quad \nabla u_3 \in SBD(\omega), \quad \bar{u} := \int_0^1 (u_1(\cdot, x_3), u_2(\cdot, x_3))\, dx_3 \in SBD(\omega),$$

$$u_\alpha(x) = \bar{u}_\alpha(x') + \left(\frac{1}{2} - x_3\right)\partial_\alpha u_3(x'), \quad J_u = (J_{\bar{u}} \cup J_{u_3} \cup J_{\nabla u_3}) \times (0,1),$$

and J_ε^f Γ-*converges to*

$$\begin{aligned} J_0(u) &= \int_\omega \left[\frac{\lambda_f \mu_f}{\lambda_f + 2\mu_f} e_{\alpha\alpha}(\bar{u}) e_{\beta\beta}(\bar{u}) + \mu_f e_{\alpha\beta}(\bar{u}) e_{\alpha\beta}(\bar{u})\right] dx' \\ &+ \frac{1}{12}\int_\omega \left[\frac{\lambda_f \mu_f}{\lambda_f + 2\mu_f} e_{\alpha\alpha}(\nabla u_3) e_{\beta\beta}(\nabla u_3) + \mu_f e_{\alpha\beta}(\nabla u_3) e_{\alpha\beta}(\nabla u_3)\right] dx' \\ &+ \underbrace{\kappa_f \mathcal{H}^1(J_{\bar{u}} \cup J_{u_3} \cup J_{\nabla u_3})}_{\textit{Griffith energy}}. \end{aligned}$$

✓ Recovery sequence (if $\bar{u}$ is “smooth” enough) :

$$u_\varepsilon(x', x_3) = u(x) - \varepsilon^2 x_3 \frac{\lambda_f}{\lambda_f + 2\mu_f} e_{\alpha\alpha}(\bar{u})(x')\vec{e}_3.$$

We assume that the original (3D) displacements are of the form

$$u(x) = u(x_1, x_3)\vec{e}_2, \qquad \text{for all } (x_1, x_3) \in I \times (-\infty, 1),$$

then for all $u \in SBV(I \times (-\infty, 1))$ with $u = 0$ in $I \times (-\infty, -1)$,

$$J^f_\varepsilon(u) = \frac{\mu_f}{2}\int_{I\times(0,1)} (|\partial_1 u|^2 + \varepsilon^{-2}|\partial_3 u|^2)\, dx_1\, dx_3 + \kappa_f \int_{J_u \cap [I\times(0,1)]} |((\nu_u)_1, \varepsilon^{-1}(\nu_u)_3)|\, d\mathcal{H}^1,$$

$$J^b_\varepsilon(u) = \frac{\mu_b}{2}\int_{I\times[-1,0]} (\varepsilon^2|\partial_1 u|^2 + |\partial_3 u|^2)\, dx_1\, dx_3 + \kappa_b \int_{J_u \cap [I\times[-1,0]]} |(\varepsilon(\nu_u)_1, (\nu_u)_3)|\, d\mathcal{H}^1,$$

and

Theorem (Leon Baldelli-B.-Bourdin-Henao-Maurini)

$J_\varepsilon = J^f_\varepsilon + J^b_\varepsilon$ "Γ-converges" to J_0, where for all $u \in SBV(I)$,

$$J_0(u) := \frac{\mu_f}{2}\int_I |u'|^2\, dx_1 + \kappa_f \#(J_u) + \frac{\mu_b}{2}\int_{I\setminus\Delta} |u|^2\, dx_1 + \underbrace{\kappa_b \mathcal{L}^1(\Delta),}_{\text{Delamination energy}}$$

with $\Delta := \{|u| > \sqrt{2\kappa_b/\mu_b}\}$.

In the general case, we conjecture that J_ε Γ-converges to

$$
\begin{aligned}
J_0(u) \quad = \quad & \int_\omega \left[\frac{\lambda_f \mu_f}{\lambda_f + 2\mu_f} e_{\alpha\alpha}(\bar{u}) e_{\beta\beta}(\bar{u}) + \mu_f e_{\alpha\beta}(\bar{u}) e_{\alpha\beta}(\bar{u}) \right] dx' \\
& + \frac{1}{12} \int_\omega \left[\frac{\lambda_f \mu_f}{\lambda_f + 2\mu_f} e_{\alpha\alpha}(\nabla u_3) e_{\beta\beta}(\nabla u_3) + \mu_f e_{\alpha\beta}(\nabla u_3) e_{\alpha\beta}(\nabla u_3) \right] dx' \\
& + \kappa_f \mathcal{H}^1(J_{\bar{u}} \cup J_{u_3} \cup J_{\nabla u_3}) + \frac{\mu_b}{2} \int_{\omega \setminus \Delta} |\bar{u}|^2 \, dx' + \kappa_b \mathcal{L}^2(\Delta),
\end{aligned}
$$

where $\Delta := \{|\bar{u}| > \sqrt{2\kappa_b/\mu_b}\} \cup \{u_3 \neq 0\}$ is the delamination set.
✓ The upper bound is true. Recovery sequence :

$$
u_\varepsilon(x', x_3) = \begin{cases} u(x) - \varepsilon^2 x_3 \frac{\lambda_f}{\lambda_f + 2\mu_f} e_{\alpha\alpha}(\bar{u})(x') \vec{e}_3 & \text{if} \quad x \in \Omega_f, \\ (x_3 + 1)(\bar{u}(x'), 0) & \text{if} \quad x \in (\omega \setminus \Delta) \times [-1, 0], \\ 0 & \text{if} \quad x \in (\Delta \times [-1, 0]) \cup \Omega_s, \end{cases}
$$

where Δ and $\bar{u}$ are suitably regularized (Chambolle).
✓ We get a too low lower bound (with a bad multiplicative constant), under additional compactness assumptions on minimizing sequences.

Thanks for your attention !