Rumour spreading in the spatial preferential attachment model

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Banff, 2016

joint work with Jeannette Janssen

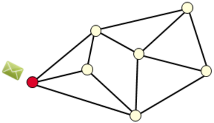
The push&pull rumour spreading protocol

[Demers, Gealy, Greene, Hauser, Irish, Larson, Manning, Shenker, Sturgis, Swinehart, Terry, Woods'87]

1. The ground is a simple connected graph.
2. At time 0, one vertex knows a rumour.
3. At each time-step $1, 2, \dots,$
   every informed vertex sends the rumour to a random neighbour (PUSH);
   and every uninformed vertex queries a random neighbour about the rumour (PULL).

Example

ROUND 0



Push-Pull Protocol

Each node contacts a random neighbor:
Node pushes the rumor (if knows);
and pulls otherwise

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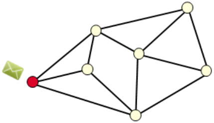

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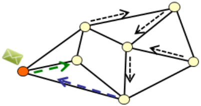

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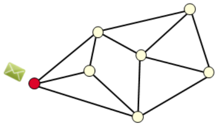

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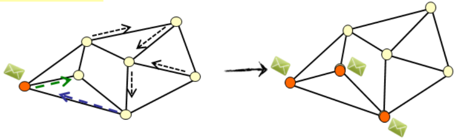

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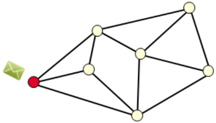

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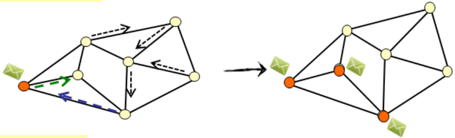
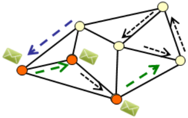


ROUND 2



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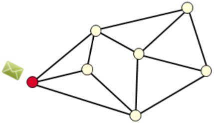

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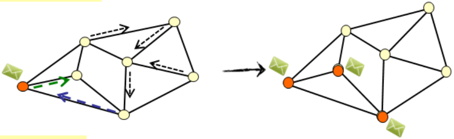

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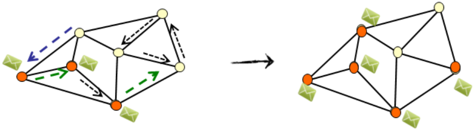

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- ✓ For a survey, see "On the push&pull rumour spreading protocol," [Acan, Collevecchio, Mehrabian, Wormald'15]

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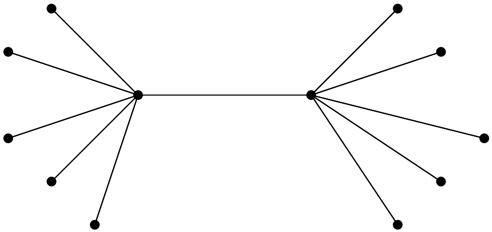

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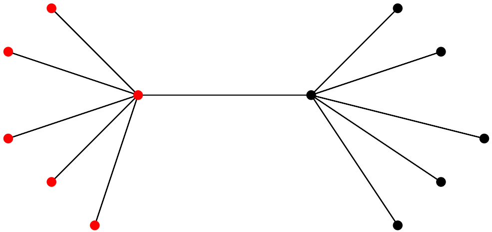

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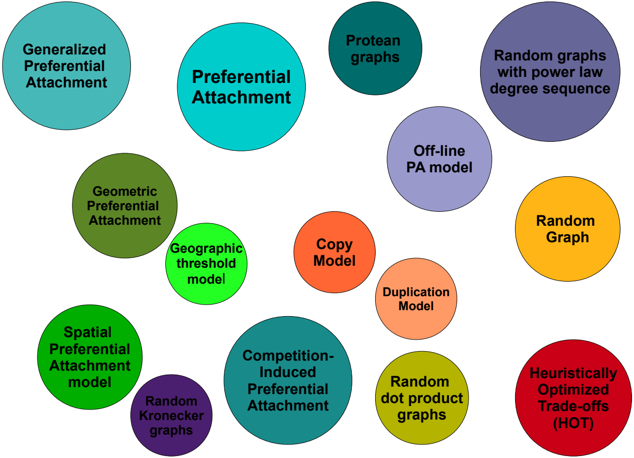

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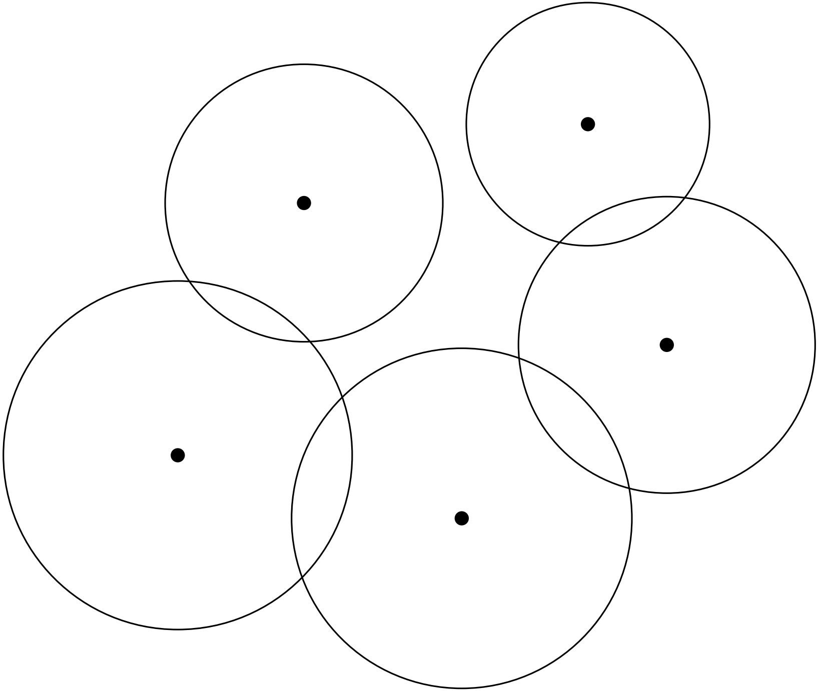
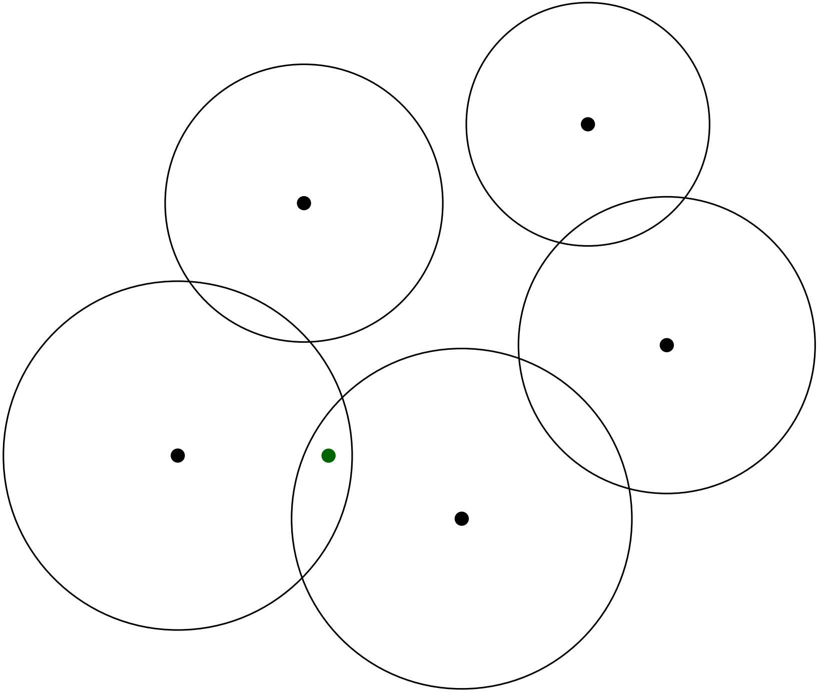
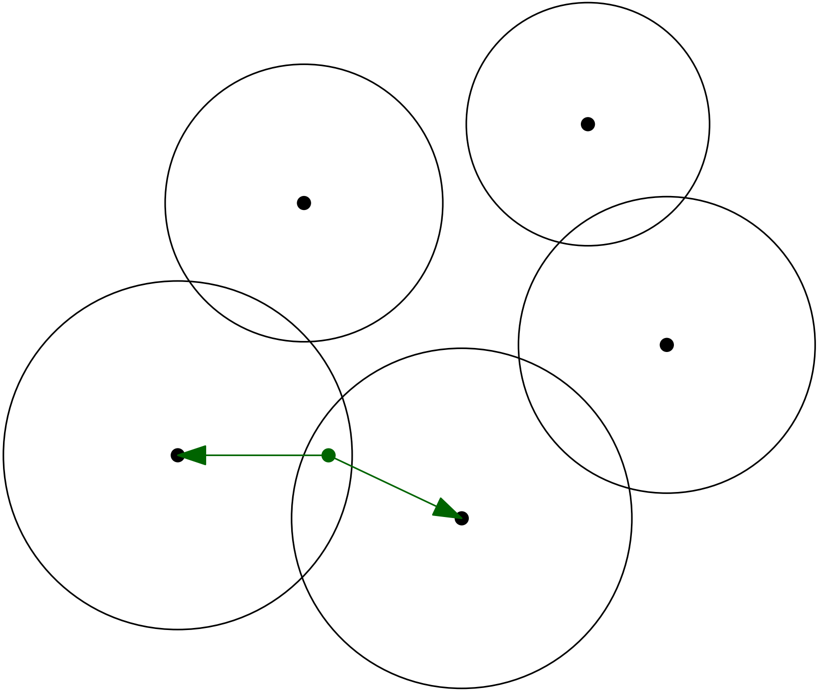
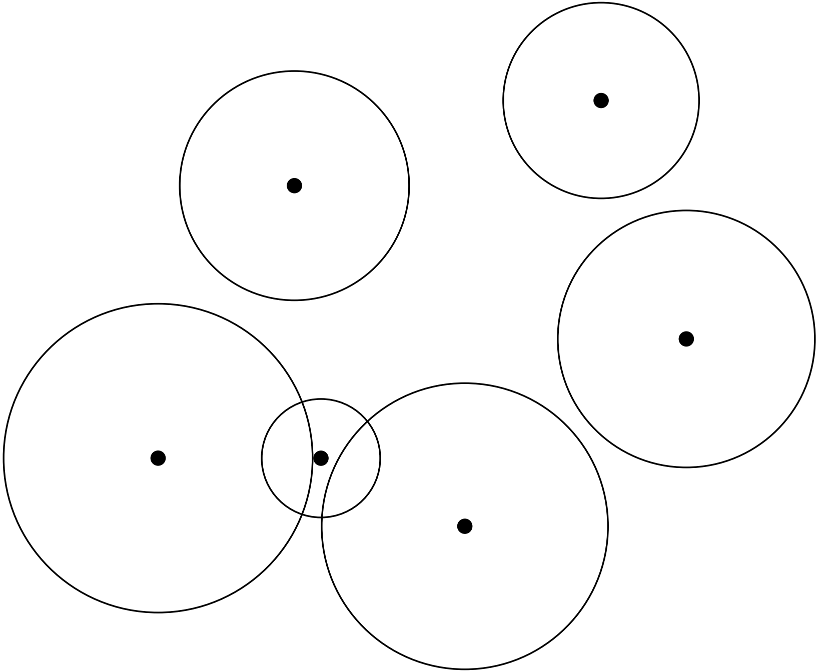
- ✓ In each time-step a new vertex is born.
- ✓ The new vertex links to an existing vertex v if it falls within the sphere of influence of v.









Implicit preferential attachment

The positions of the vertices are uniformly random,
the space has volume 1, so the probability that v receives a link at time t is

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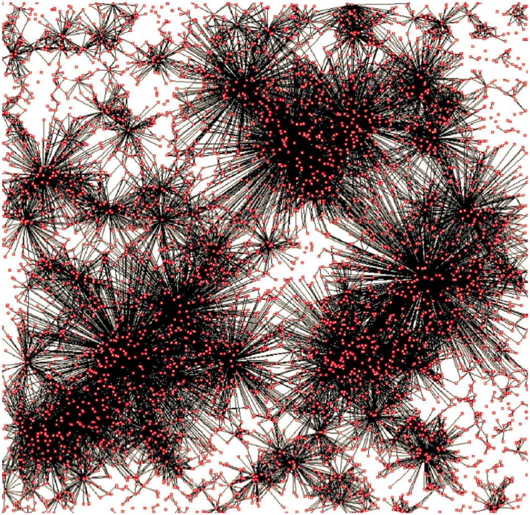
- ✓ Original model has parameter p, we consider $p = 1$ only.

SPA model: example in 2D



5000 vertices, $A_1 = A_2 = 1$ [Cooper,Frieze,Prałat'14]

Known results

- ✓ **Power law degree distribution.** If $A_1 < 1$, the in-degree distribution has a power law tail with exponent $1 + \frac{1}{A_1}$
  [Aiello,Bonato,Cooper,Janssen,Prałat'08]
- ✓ **Sparse graph.** If $A_1 < 1$, a.a.s. the average out-degree is $\frac{A_2}{1-A_1}$ [Aiello,Bonato,Cooper,Janssen,Prałat'08]
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- ✓ **Good clustering**
- ✓ **Large maximum degree.** A.a.s. maximum total degree is $n^{\Omega(1)}$ [Aiello,Bonato,Cooper,Janssen,Prałat'08]
- ✓ **Not an expander.** A.a.s. the minimum bisection has size $o(n)$ [Cooper,Frieze,Prałat'14]

Our results

Consider the giant component of the undirected underlying graph generated by the SPA model;

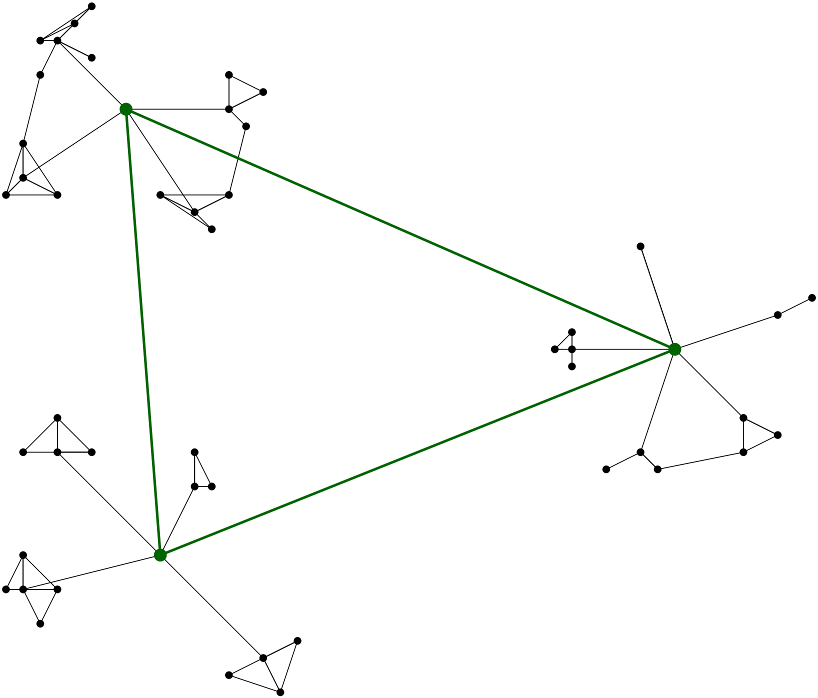
Theorem (upper bound for the diameter)

If A_2 is sufficiently large, a.a.s. the graph has diameter $O(\log^2 n)$.

Note: Result of Cooper-Frieze-Prałat does not apply, as they consider directed paths only.

Theorem (lower bound for rumour spreading)

A.a.s. it takes $n^{\Omega(1)}$ rounds to spread the rumour.



Proof sketch of the upper bound for diameter

Say n is a power of 2

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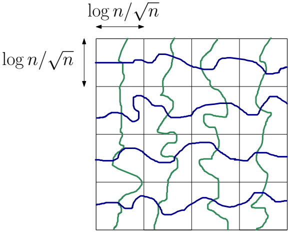
   - $G_{n/2}$ contains a random geometric graph.
   - By known results on RGG's, any point not in the giant of $G_{n/2}$ is within $\log n/\sqrt{n}$ Euclidean distance to some vertex in the giant of $G_{n/2}$. [Ganesan'13]



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   - By known results on stretch factor of RGG's, this leads to a path of length $O(\log n)$ from v to some vertex in the giant of $G_{n/2}$. [Bradonjic, Elsasser, Friedrich, Sauerwald, Stauffer'13]

Upper bound for diameter (the catch)

$$G_n \supseteq \bigcup_{i=1}^{n} R_i,$$

where each R_i is a RGG.

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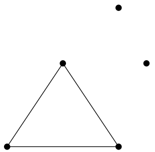

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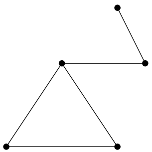

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By [Penrose'03], 99% of vertices lie in the giant.

Theorem (Janssen, M'15)

In the SPA model graph with $p = 1$ and A_2 sufficiently large, a.a.s. 99% of vertices are within distance $O(\log^2 n)$ of each other, in dimension 2.

Further questions. . .

Proof sketch of the lower bound for rumour spreading

1. Categorize the edges into *short* and *long*, and prove that no *long* edge is used during the first $n^{O(1)}$ rounds.
2. Vertices that are born late, have small spheres of influence

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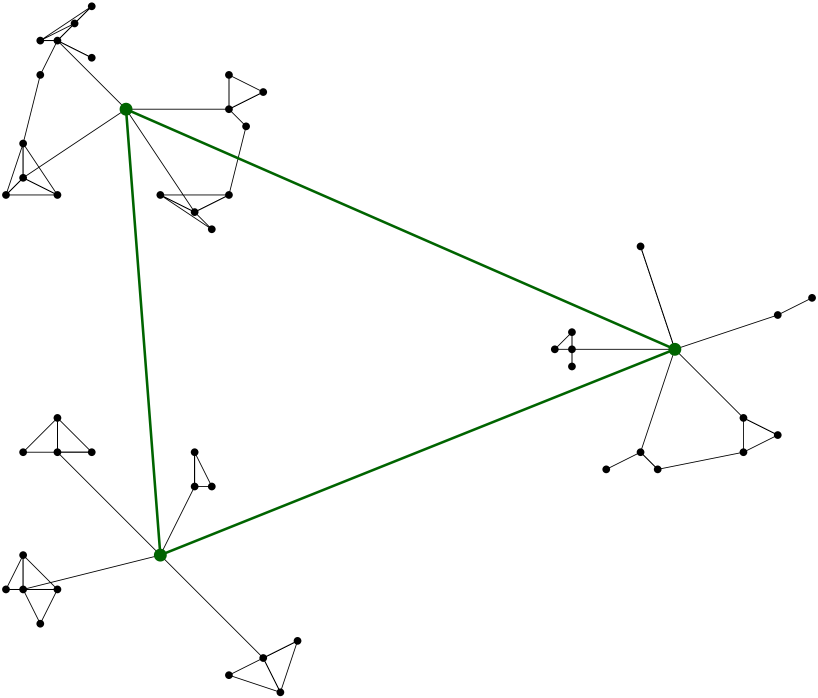
Proof technique: Old and new concentration inequalities for vertices' degrees.



Wrap up

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Theorem (Janssen, M'15)

Suppose $pA_1 < 1$. If rumour starts from a random vertex, a.a.s. after n^α rounds, number of informed vertices is $o(n)$.

$\alpha = \frac{pA_1(1-pA_1)}{(3+pA_1)\times\text{dimension}+1-pA_1}$