Regularity and Decay Estimates of the Navier-Stokes Equations

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The incompressible Navier-Stokes equations are given by

$$u_t + u \cdot \nabla u - \Delta u + \nabla p = 0, \tag{1a}$$

$$\nabla \cdot u = 0, \tag{1b}$$

where u is the velocity field and p is the pressure.

For a divergence-free vector field $u_0 \in L^2$, there exists a weak solution satisfying the energy inequality (Leray):

$$\|u(t)\|_{L^2}^2 + 2\int_0^t \|\nabla u(s)\|_{L^2}^2 \, ds \leq \|u_0\|_{L^2}^2 .$$

This implies that

$$\liminf_{t\to\infty} \|\nabla u(t)\|_{L^2} = 0.$$

To show this, for $\epsilon > 0$ arbitrary, choose t large so that

$$\frac{1}{t}\|u_0\|_{L^2}^2 < 2\epsilon.$$

Then, the energy inequality implies

$$\frac{1}{t}\int_0^t \|\nabla u(s)\|_{L^2}^2 \, ds \le \frac{1}{2t}\|u_0\|_{L^2}^2 < \epsilon.$$

By Pigeonhole principle, there exists $t_0 \in (0, t)$ such that

$$\|\nabla u(t_0)\|_{L^2}^2 < \epsilon.$$

On a smooth bounded domain with Dirichlet boundary condition, weak solutions decay exponentially.

From the energy inequality

$$\frac{d}{dt}\left\|u(t)\right\|_{L^2(\Omega)}^2 \leq -2\left\|\nabla u(t)\right\|_{L^2(\Omega)}^2$$

and Poincaré's inequality

$$\left\|u(t)\right\|_{L^2} \leq C\left\|\nabla u(t)\right\|_{L^2},$$

we have

$$\frac{d}{dt}\left\|u(t)\right\|_{L^2}^2 \leq -C\left\|u(t)\right\|_{L^2}^2 \quad \Longrightarrow \quad \left\|u(t)\right\|_{L^2} \leq \left\|u_0\right\|_{L^2} e^{-Ct}.$$

On the whole space $\mathbb{R}^3$, Schonbek (1985) proved that for divergence-free $u_0 \in L^1 \cap L^2$, weak solutions satisfy the following algebraic decay property:

$$\|u(t)\|_{L^2} \leq C(t+1)^{-\frac{1}{4}}$$

using Fourier splitting method.

(1) L^2: Schonbek (1985), Wiegner (1987)

(2) H^s: Schonbek-Wiegner (1996)

This talk consists of 2 parts.

1. The incompressible Navier-Stokes equations

(1) Mild solution of the Navier-Stokes equations

(2) Gevrey regularity of mild solutions

(3) Decay estimates of weak solutions

(4) Log-Lipschitz regularity of the velocity and Hölder regularity of the flow map

2. Dissipative equations with analytic nonlinearity

MILD SOLUTION

A mild solution $u(t,x)$ of the Navier-Stokes equations satisfies the integral equation

$$u(t) = e^{t\Delta}u_0 - \int_0^t \left[e^{(t-s)\Delta}\mathbb{P}\nabla \cdot (u \otimes u)(s)\right] ds := e^{t\Delta}u_0 - B(u,u),$$

where

1. $u(t,x) \in C([0,T];X)$;
2. X is a Banach space on which the heat semi-group

$$\left\{e^{t\Delta} : t \geq 0\right\}$$

   is strongly continuous.

Fixed Point Lemma. Let X be a Banach space and $B : X \times X \to X$ a bilinear operator, such that for any $u, v \in X$,

$$\|B(u,v)\|_X \leq \eta \|u\|_X \|v\|_X.$$

Then, for any $u_0 \in X$ with smallness condition

$$4\eta \|u_0\|_X \leq 1,$$

the equation

$$u(t) = e^{t\Delta} u_0 - \int_0^t \left[e^{(t-s)\Delta} \mathbb{P} \nabla \cdot (u \otimes u)(s) \right] ds := e^{t\Delta} u_0 - B(u,u) \tag{2}$$

has a unique global in time solution $u \in C([0,\infty); X)$.

How to find function spaces solving (2)? Roughly,

$$
\begin{aligned}
u \in W^{s,p} &\implies u \otimes u \in W^{2s-\frac{3}{p},p} \\
&\implies \nabla \cdot (u \otimes u) \in W^{2s-\frac{3}{p}-1,p} \\
&\implies \int_0^t \left[e^{(t-s)\Delta} \mathbb{P} \nabla \cdot (u \otimes u)(s) \right] ds \in W^{2s-\frac{3}{p}-1+2,p}.
\end{aligned}
$$

$$
s = 2s - \frac{3}{p} - 1 + 2 \implies s = \frac{3}{p} - 1
$$

Function spaces associated with (2) correspond to the scaling invariance property of the equation:

$$u_\lambda(t,x) = \lambda u(\lambda^2 t, \lambda x), \quad p_\lambda(t,x) = \lambda^2 p(\lambda^2 t, \lambda x) \quad \forall \lambda > 0$$

A space X is called a critical space (for initial data) if its norm is invariant under the scaling

$$f(x) \mapsto \lambda f(\lambda x).$$

Homogeneous critical spaces in 3D

$$\dot{H}^{\frac{1}{2}}, \quad L^3, \quad \dot{W}^{\frac{3}{p}-1,p}, \quad \dot{B}^{\frac{3}{p}-1}_{p<\infty,q}, \quad \mathrm{BMO}^{-1}, \quad \dot{B}^{-1}_{\infty,\infty}$$

Mild Solutions of the Navier-Stokes Equations in Critical Spaces

(1) $\dot{H}^{\frac{1}{2}}$: Fujita–Kato (1964)

(2) L^3: Kato (1984), Furioli–Lemarié-Rieusset–Terraneo (2000)

(3) $\dot{B}^{\frac{3}{p}-1}_{p<\infty,q}$: Cannone–Meyer–Planchon (1993), Cannone–Planchon (1996), Chemin (1999)

(4) BMO^{-1}: Koch–Tataru (2001)

(5) Fourier Space: Le Jan–Sznitman (1997), Lei–Lin (2011)

We note that the Navier-Stokes equation is ill-posed in $\dot{B}^{-1}_{\infty,\infty}$ (Bourgain-Pavlović (2008))

The space analyticity radius yields a Kolmogorov type length scale encountered in turbulence theory; at this length scale the viscous effects and the (nonlinear) inertial effects are roughly comparable.

Below this length scale the Fourier spectrum decays exponentially. This fact can be used to show that the Galerkin approximations converge exponentially fast.

Example: Let $u(x) = \sum_{k \in \mathbb{Z}^d} u_k e^{i(x \cdot k)}$ on $\mathbb{T}^d$. Let δ be the size of the analyticity radius in $L^2\left(\mathbb{T}^d\right)$. Then,

$$|u_k| \lesssim e^{-\delta|k|} \quad \text{(Paley-Weiner Theorem).}$$

Analyticity radius is used to establish geometric regularity criteria for the Navier-Stokes equations, and to measure the spatial complexity of fluid flow. (Kukavica (1996,1999), Grujic (2001))

We cannot use Cauchy-Kovalevski theorem to show analyticity of the Navier-Stokes equations due to the non-local term ∇p,

$$\nabla p = \nabla(-\Delta)^{-1}\nabla\cdot (u \cdot \nabla u)\,.$$

Instead, we will use the fact that mild solutions are perturbation of solutions to the heat equation.

SPATIAL ANALYTICITY IN CRITICAL SPACES

1. Convergence of Taylor series

$$\left\|\nabla^k u\right\|_{\mathrm{BMO}^{-1}} \leq \frac{C^k k!}{(\sqrt{t})^k}$$

Germain-Pavlović-Staffilani (2007)

$\Longrightarrow$ Koch-Tatatu solution is analytic.

2. Analytic Extension in L^p, $p \geq 3$

$$U_t + U \cdot \nabla U + \nabla P - \Delta U = 0, \quad U(x+iy,0) = u_0(x) \in L^p$$

Grujic- Kukavica (1998), Guberović (2010)

$\Longrightarrow$ L^p solutions (including Kato solution) are analytic.

GEVREY REGULARITY

A function f is Gevrey regular of order α if there exist M and R such that

$$\left\|\nabla^k f\right\|_{L^\infty} \leq M^k \frac{(k!)^\alpha}{R^k}, \quad \forall k \in \mathbb{N}.$$

Gevrey regularity of order 1 is analytic by the Cauchy estimates.

In the study of evolution equations, a function f is said to be Gevrey regular of order α in a Banach space X if

$$e^{\tau \sqrt{-\Delta}^\alpha} f \in X$$

for some $\tau > 0$.

We note that $e^{\tau\sqrt{-\Delta}} f \in X$ implies that f is analytic in X (by Cauchy estimates), with the radius of analyticity proportional to τ.

Gevrey Regularity approaches to the Navier-Stokes Equations

(1) H^1: Foias–Temam (1989), Oliver–Titi (2000).

(2) $\dot{H}^{\frac{1}{2}}$: Lemarié-Rieusset (2002) : $|\xi|^{\frac{1}{2}} \sup_{t>0} e^{\sqrt{t}|\xi|} |\hat{u}(t,\xi)| \in L^2$.

$\Longrightarrow$ Fujita–Kato solution is analytic.

Lemarié-Rieusset: Mild solutions are analytic for $t > 0$.

Let X be a critical space. If u is a mild solution,

$$u(t) = e^{t\Delta}u_0 - \int_0^t e^{(t-s)\Delta}\mathbb{P}\nabla\cdot(u\otimes u)(s)ds \ \in X,$$

the same is true for $e^{\sqrt{t}\Lambda_1}u \in X$.

(1) The exponential operator $e^{\sqrt{t}\Lambda_1}$ is quantified by Λ_1 whose symbol is given by $|\xi|_1 = |\xi_1| + |\xi_2| + |\xi_3|$ rather than the usual $\Lambda = (-\Delta)^{\frac{1}{2}}$.

(2) The inverse operator $e^{-\sqrt{t}\Lambda_1}$ is a Fourier multiplier, which is the product of one dimensional Poisson kernels. The L^1 norm of this kernel is bounded by a constant independent of t.

Recent developments in the Navier-Stokes problem, Lemarié-Rieusset (2002)

To prove $e^{\sqrt{t}\Lambda_1} u \in X$, we need to show that the bilinear term

$$B(u,u) = \int_0^t e^{(t-s)\Delta} \mathbb{P}\nabla \cdot (u \otimes u)(s) ds$$

is bounded in X under the action of $e^{\sqrt{t}\Lambda_1}$.

Let $U(t) = e^{\sqrt{t}\Lambda_1} u$.

$$e^{\sqrt{t}\Lambda_1} B(u,u) = e^{\sqrt{t}\Lambda_1} \int_0^t \left[e^{(t-s)\Delta} \mathbb{P}\nabla \cdot (e^{-\sqrt{s}\Lambda_1} U \otimes e^{-\sqrt{s}\Lambda_1} U)(s) \right] ds$$

$$= \int_0^t \left[e^{(\sqrt{t}-\sqrt{s})\Lambda_1} e^{\frac{1}{2}(t-s)\Delta} e^{\frac{1}{2}(t-s)\Delta} e^{\sqrt{s}\Lambda_1} \mathbb{P}\nabla \cdot (e^{-\sqrt{s}\Lambda_1} U \otimes e^{-\sqrt{s}\Lambda_1} U)(s) \right] ds$$

$$\int_0^t \left[e^{\frac{1}{2}(t-s)\Delta}\mathbb{P}\nabla \cdot e^{\sqrt{s}\Lambda_1}(e^{-\sqrt{s}\Lambda_1}U \otimes e^{-\sqrt{s}\Lambda_1}U)(s)\right]ds$$

We introduce the bilinear operator B_s of the form

$$B_s(f,g) = e^{\sqrt{s}\Lambda_1}(e^{-\sqrt{s}\Lambda_1}fe^{-\sqrt{s}\Lambda_1}g).$$

Roughly,

$$B_s(f,g) \simeq f\left(e^{-\sqrt{s}\Lambda_1}g\right) + \left(e^{-\sqrt{s}\Lambda_1}f\right)g \simeq fg.$$

Lemarié-Rieusset proved that

$$B_s(f,g) = K_{\alpha_1} \otimes K_{\alpha_2} \otimes K_{\alpha_3}(fg) \ \simeq \ fg,$$

where K_{α_i} are linear combinations of the Poisson kernels and identity operators.

$$U(t) = e^{\sqrt{t}\Lambda_1}u, \quad U(t) \simeq e^{\frac{1}{2}t\Delta}u_0 + \int_0^t \left[e^{\frac{1}{2}(t-s)\Delta}\mathbb{P}\nabla \cdot (U \otimes U)(s)\right]ds.$$

GEVREY REGULARITY IN CRITICAL SPACES

1. $\left\{f \in \mathscr{S}' : |\xi|^2 \left|\widehat{f}(\xi)\right| \in L^\infty\right\}$
   (1) Le Jan–Sznitman (1997)
   (2) Analyticity: Lemarié-Rieusset (2002)

2. $\dot{B}^{\frac{3}{p}-1}_{p<\infty,\infty}$
   (1) Cannone–Meyer–Planchon (1993), Cannone–Planchon (1996), Chemin (1999)
   (2) Analyticity: B–Biswas–Tadmor (2012)

3. $\dot{B}^{-1}_{\infty,q} \cap \dot{B}^{0}_{3,\infty}$, $1 \leq q < \infty$
   (1) Existence and Analyticity: B–Biswas–Tadmor (2012)

4. $\left\{f \in \mathscr{S}' : |\xi|^{-1} \left|\widehat{f}(\xi)\right| \in L^1\right\}$
   (1) Lei–Lin (2011)
   (2) Analyticity: B. (2015)

DECAY ESTIMATES OF WEAK SOLUTIONS

Gevrey regularity enables us to obtain decay estimates:

$$\left\|\nabla^k u(t)\right\|_X = \left\|\nabla^k e^{-\sqrt{t}\Lambda_1} e^{\sqrt{t}\Lambda_1} u(t)\right\|_X \le C_k t^{-\frac{k}{2}} \left\|e^{\sqrt{t}\Lambda_1} u(t)\right\|_X .$$

If $\|u_0\|_X$ is sufficiently small, $\left\|e^{\sqrt{t}\Lambda_1} u(t)\right\|_X$ is uniformly bounded.

$$\left\|\nabla^k u(t)\right\|_X \le C_k t^{-\frac{k}{2}}, \quad k \in \mathbb{N}.$$

If $\|u_0\|_Y$ is large, but $\liminf_{t\to\infty} \|u(t)\|_X = 0$, after a certain transient time $t_0 > 0$

$$\left\|\nabla^k u(t)\right\|_X \le C_k (t - t_0)^{-\frac{k}{2}}, \quad k \in \mathbb{N}.$$

Theorem (B–Biswas–Tadmor (2012)): For any initial data $u_0 \in L^2$ with $\omega_0 \in L^1$, there exists a time $t_0 > 0$ such that weak solutions decay as

$$\left\|\nabla^k u(t)\right\|_{L^p} \leq C_k \|u(t_0)\|_{\dot{B}^{\frac{3}{p}-1}_{p,p}} (t-t_0)^{-\frac{k-s_0}{2}}, \quad k > s_0 = \frac{3}{p} - 1, \quad p < 2.$$

(1) This is decay rate in L^p-based spaces with $p < 2$.

(2) The main idea is to find small critical norm only using the L^2 norm of u and the L^1 norm of the vorticity $\omega = \nabla \times u$.

(3) We cannot express $t_0 > 0$ explicitly.

PROOF

1. L^2-level: $u_0 \in L^2 \implies \|u(t)\|_{L^2}^2 + \int_0^t \|\nabla u(s)\|_{L^2}^2 ds \leq \|u_0\|_{L^2}^2$

$$\implies \liminf_{t\to\infty} \|\nabla u(t)\|_{L^2} = 0.$$

2: L^1-level: $\omega_0 = \nabla \times u_0 \in L^1$

$$\omega_t + u \cdot \nabla\omega - \Delta\omega = \omega\nabla u \quad \implies \quad \|\omega(t)\|_{L^1} \leq C.$$

3. Interpolation:

$$\liminf_{t\to\infty} \|u(t)\|_{\dot{B}^{\frac{3}{p}-1}_{p,p}} = 0, \quad p < 2.$$

4. Decay rates: from the relation between Besov spaces and Sobolev spaces

$$\dot{B}^{\frac{3}{p}-1}_{p,p} \subset \dot{W}^{\frac{3}{p}-1,p} \quad \text{for } p < 2$$

Osgood's lemma. Let ρ be a positive measurable, γ be a positive and locally integrable function, and μ be a continuous increasing function. Suppose that for a positive real number a, ρ satisfies

$$\rho(t) \leq a + \int_0^t \gamma(s)\mu(\rho(s))ds.$$

Let

$$\Phi(x) = \int_x^1 \frac{1}{\mu(r)}dr.$$

Then, we have

$$-\Phi(\rho(t)) + \Phi(a) \leq \int_0^t \gamma(s)ds.$$

Furthermore, if $a = 0$, then ρ is identically zero.

Theorem. Let X be a Banach space, Ω an open set of X, I an open interval of $\mathbb{R}$ and $(0, x_0) \in I \times \Omega$. Let F be a function in $L^1_{\mathrm{loc}}(I, C_\mu(\Omega))$ with

$$\int_0^1 \frac{1}{\mu(r)} dr = \infty.$$

Then, there exists an interval J such that $0 \in J \subset I$ and the equation

$$x(t) = x_0 + \int_0^t F(s, x(s)) ds$$

has a unique continuous solution on J (by Picard iteration and Osgood lemma)

(1) $\mu(r) = r$ when F is Lipschitz.

(2) $\mu(r) = r(-\ln r)^\beta$, $0 < \beta \leq 1$ when F is Log-Lipschitz.

Zuazua (2002) proved Log-Lipschitz regularity and uniqueness of the flow for a field in

$$L^1\left(0,T;W^{\frac{3}{p}+1,p}\right).$$

The flow map η of the Navier-Stokes equations:

$$\partial_t\eta(t,x) = u\left(t,\eta(t,x)\right), \quad \eta(0,x) = x.$$

We note that mild solutions of the Navier-Stokes equations gain almost 2 derivatives when integrated in time:

$$u_0 \in \dot{W}^{\frac{3}{p}-1,p} \implies u \in L^1\left(0,T;\dot{W}^{\frac{3}{p}+1-\epsilon,p}\right).$$

Log-Lipschitz regularity

Chemin–Lerner (1995): for $u_0 \in H^{\frac{1}{2}}$, there exists $T > 0$ such that the exists a unique solution $u \in C([0,T]; H^{\frac{1}{2}})$ such that

(1) u is Log-Lipschitz

$$\int_0^T \sup_{|x-y|<\frac{1}{2}} \frac{|u(t,x)-u(t,y)|}{|x-y|\,(-\ln|x-y|)^{\beta}} dt < C\|u_0\|_{H^{\frac{1}{2}}}, \quad \beta > \frac{1}{2}$$

(2) $\eta \in C([0,T] : C^{\alpha}) \;\; \forall\, \alpha \in (0,1)$

(3) Chemin-Lerner space:

$$\widetilde{L^r_T}\dot{B}^s_{p,q} = \left\{ f \in \mathcal{S}_h' : \; \|f\|_{\widetilde{L^\rho_T}\dot{B}^s_{p,q}} = \left\| 2^{js} \|\Delta_j f\|_{L^r_T L^p} \right\|_{l^q(\mathbb{Z})} < \infty \right\}$$

This space gains two derivatives when $r = 1$.

Theorem (B–Cannone (2016)). There exists a constant $\epsilon > 0$ such that for all $u_0 \in L^1 \cap H^{\frac{1}{2}}$ with the smallness condition

$$\|u_0\|_{\dot{H}^{\frac{1}{2}}} \leq \epsilon,$$

there exists a global solution u satisfying the following Log-Lipschitz regularity:

$$\|u\|_{LL_\beta} := \int_0^\infty \sup_{|x-y|<\frac{1}{2}} \frac{|u(t,x) - u(t,y)|}{|x-y| \left(-\ln|x-y|\right)^\beta} dt \leq C_\beta \left(\|u_0\|_{L^1} + \|u_0\|_{H^{\frac{1}{2}}} \right)$$

for $\beta > \frac{1}{2}$.

There exists a unique flow map η satisfying the following Hölder regularity:

(i) $\beta \in \left(\frac{1}{2}, 1\right)$: for $|x-y| \leq e^{-(1-\beta)\|u\|_{LL_\beta}^{1-\beta}}$,

$$|\eta(t,x) - \eta(t,y)| \leq |x-y|^{\alpha}, \quad \forall \alpha \in (0,1). \tag{3}$$

(ii) $\beta = 1$: for $|x-y| < \dfrac{1}{2}$,

$$|\eta(t,x) - \eta(t,y)| \leq |x-y|^{\gamma}, \quad \gamma = e^{-\|u\|_{LL_1}}. \tag{4}$$

In particular, the Hölder exponent γ has a lower bound when $\beta = 1$:

$$\gamma \geq C e^{-\left(\|u_0\|_{L^1} + \|u_0\|_{H^{\frac{1}{2}}}\right)}$$

1. We control low frequency part by $u_0 \in L^1 \cap L^2$ and high frequency part by small $u_0 \in \dot{H}^{\frac{1}{2}}$ to show Log-Lipschitz regularity.

2. The existence of a continuous solution η follows by ODE theorem with

$$\mu(r) = r(-\ln r)^{\beta}, \quad \frac{1}{2} < \beta \leq 1.$$

We restrict the range of $\beta \leq 1$ to satisfy the condition

$$\int_0^1 \frac{1}{\mu(r)} dr = \infty.$$

Theorem (B–Cannone (2016)). There exists a constant $\epsilon_0 > 0$ such that for all $u_0 \in \dot{H}^{\frac{1}{2}}$ with $\|u_0\|_{\dot{H}^{\frac{1}{2}}} \leq \epsilon_0$, there exists a global in time solution u satisfying the following Log-Lipschitz regularity: for any $k \in \mathbb{N}$

$$\int_0^T \sup_{|x-y|<\frac{1}{2}} \frac{t^{\frac{k}{2}} \left|\nabla^k \left(u(t,x) - u(t,y)\right)\right|}{|x-y| \left(-\ln|x-y|\right)^{\beta}} dt \leq C_k \ln T \|u_0\|_{\dot{H}^{\frac{1}{2}}} \quad \text{for } \beta > 1.$$

This theorem implies that the Log-Lipschitz regularity holds almost globally:

$$\int_0^T \sup_{|x-y|<\frac{1}{2}} \frac{t^{\frac{k}{2}} \left|\nabla^k \left(u(t,x) - u(t,y)\right)\right|}{|x-y| \left(\ln|x-y|\right)^{\beta}} dt \leq 1, \quad T \sim e^{\frac{1}{\epsilon_0}}.$$

Compared to the Euler equations,

(1) Yudovich (1963) proved Hölder regularity of the flow map of the 2D Euler equations with initial vorticity in L^∞. But, Hölder regularity of the flow map is decreasing in time.

This losing regularity does not occur to the Navier-Stokes equations when initial data are small in critical spaces.

(2) Beale-Kato-Majda blowup criterion (1984) is obtained by dealing with the Biot-Savart kernel near the origin and infinity separately in the real variables.

We prove our results by analyzing u near the origin and infinity separately in the Fourier variables.

We only show how to obtain Hölder regularity of the flow map when $\beta = 1$. Let

$$\chi(t) := \chi(t,x,y) = \big|\eta(t,x) - \eta(t,y)\big|,$$
$$\|u\|_{LL} := \int_0^\infty \sup_{|x-y|<\frac{1}{2}} \frac{\big|u(t,x) - u(t,y)\big|}{|x-y|\left(-\ln|x-y|\right)} dt.$$

From the equation of η, we have the following inequality

$$\chi^{'}(t) \leq \left|u(t,\eta(t,x)) - u(t,\eta(t,y))\right|. \tag{5}$$

We rewrite (5) as

$$-\frac{d}{dt}\left[\ln\left(-\ln\chi(t)\right)\right] = \frac{\chi^{'}(t)}{\rho(t)\left(-\ln\chi(t)\right)} \leq \frac{\left|u(t,\eta(t,x)) - u(t,\eta(t,y))\right|}{\chi(t)\left(-\ln\chi(t)\right)}. \tag{6}$$

Integrating (6) in time, we have

$$-\ln\left(-\ln\chi(t)\right)+\ln\left(-\ln\chi(0)\right)\leq\int_0^t\frac{|u(s,\eta(s,x))-u(s,\eta(s,y))|}{\chi(s)\left(-\ln\chi(s)\right)}dt\leq\|u\|_{LL}. \qquad (7)$$

By taking a double exponential to (7), we have

$$\chi(t)\leq\chi(0)^{e^{-\|u\|_{LL}}}$$

as long as $\chi(s)<\frac{1}{2}$ for $0<s<t$ and this holds for $|x-y|<\frac{1}{2}$.

Therefore, we obtain that

$$|\eta(t,x)-\eta(t,y)|\leq|x-y|^{e^{-\|u\|_{LL}}}.$$

ANALYTIC NONLINEARITY: HOMOGENEOUS CASE

We consider dissipative equations of the form

$$u_t - \Delta u = \nabla F(u), \quad F(z_1, \cdots, z_d) = \sum_{|\alpha|=n} a_\alpha z^\alpha, \quad n \geq 2.$$

and homogeneous Sobolev/Gevrey spaces

$$\dot{W}^{s,p} = \left\{ u : R^d \to R^m : \|u\|_{\dot{W}^{s,p}} = \|\Lambda^s u\|_{L^p} < \infty \right\},$$

$$\dot{G}(t,s,p) = \left\{ u \in \dot{W}^{s,p} : \ \|u\|_{\dot{G}(t,s,p)} = \left\| e^{\sqrt{t}\Lambda_1} u \right\|_{\dot{W}^{s,p}} < \infty \right\}.$$

Gevrey Regularity in Energy spaces ($p = 2$)

Ferrari–Titi (1998), Cao-Rammaha–Titi (1999)

1. Critical spaces: $\dot{W}^{s_0,p}$, $s_0 = \dfrac{d}{p} - \dfrac{1}{n-1} > 0$

2. Fractional product rule

$$\|\Lambda^s(fg)\|_{L^p} \lesssim \|\Lambda^s f\|_{L^{p_1}} \|g\|_{L^{q_1}} + \|f\|_{L^{p_2}} \|\Lambda^s g\|_{L^{q_2}}\,, \quad \frac{1}{p} = \frac{1}{p_i} + \frac{1}{q_i}$$

$$\left\|e^{\sqrt{t}\Lambda_1}\Lambda^s(fg)\right\|_{L^p} \lesssim \left\|e^{\sqrt{t}\Lambda_1}\Lambda^s f\right\|_{L^{p_1}} \left\|e^{\sqrt{t}\Lambda_1} g\right\|_{L^{q_1}} + \left\|e^{\sqrt{t}\Lambda_1} f\right\|_{L^{p_2}} \left\|e^{\sqrt{t}\Lambda_1}\Lambda^s g\right\|_{L^{q_2}}.$$

Theorem (B-Biswas (2015)): For $\|u_0\|_{\dot{W}^{s_0,p}} < \epsilon \ll 1$, $p < 2$

(1) Gevrey regularity: $e^{\sqrt{t}\Lambda_1} u(t) \in \dot{W}^{s_0,p}$.

(2) Decay: $\left\|\nabla^k u(t)\right\|_{L^p} \leq C_k t^{-\frac{1}{2}(k-s_0)}$, $\quad k > s_0$.

ANALYTIC NONLINEARITY: INHOMOGENEOUS CASE

We consider dissipative equations of the form

$$u_t - \Delta u = F(u), \quad F(z) = \sum_{n \in \mathbb{Z}^d} a_n z^n$$

with its majorizing function $\mathscr{F}(r) = \sum_n |a_n| r^{|n|}$ converging $\forall r > 0$.

<u>Example:</u> $u_t - \Delta u = \sin u$

We consider the inhomogeneous Sobolev/Gevrey spaces

$$W^{s,p} = \left\{ u : R^d \to R^m : \|u\|_{W^{s,p}} = \|(1+\Lambda)^s u\|_{L^p} < \infty \right\}$$

$$G(t,s,p) = \left\{ u \in W^{s,p} : \ \|u\|_{G(t,s,p)} = \left\| e^{\sqrt{t}\Lambda_1} u \right\|_{W^{s,p}} < \infty \right\}.$$

1. Existence of a solution: $W^{s,p}$ is a Banach algebra when $s > \frac{d}{p}$

$$\|fg\|_{W^{s,p}} \leq C \|f\|_{W^{s,p}} \|g\|_{W^{s,p}} \quad s > \frac{d}{p}.$$

2. $G(t,s,p)$ is a Banach algebra when $s > \frac{d}{p}$:

$$\|fg\|_{G(t,s,p)} \leq C \|f\|_{G(t,s,p)} \|g\|_{G(t,s,p)} .$$

Theorem (B-Biswas (2015)): Let $\|u_0\|_{W^{s,p}} < C$, with $s > \frac{d}{p}$. Then, there exists a time $T > 0$ and a solution $u \in L^\infty(0,T; W^{s,p})$ such that

$$\sup_{t \in (0,T]} \|u(t)\|_{G(t,s,p)} < \infty.$$

Thank you for your attention