The three-dimensional free-boundary Euler equations with surface tension

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Joint work with David G. Ebin (SUNY at Stony Brook).

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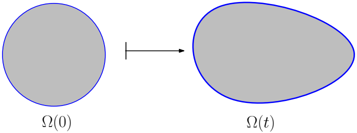

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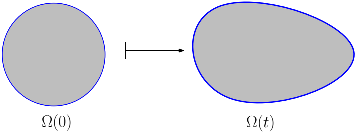

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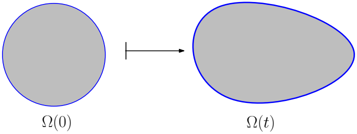

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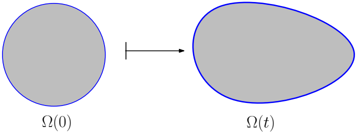

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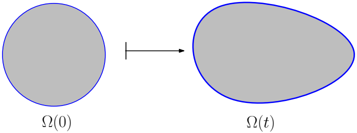

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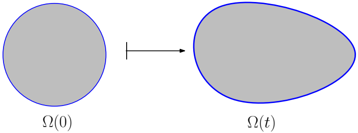

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Terminology: fluids = incompressible inviscid fluids.

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\begin{cases}
\dfrac{\partial u}{\partial t} + (u \cdot \nabla) u = -\nabla p & \text{in } \bigcup_{0 \le t \le T} \{t\} \times \Omega(t), \quad (1a)\\
\operatorname{div}(u) = 0 & \text{in } \Omega(t), \quad (1b)\\
p = \kappa \mathcal{A} & \text{on } \partial\Omega(t), \quad (1c)\\
\partial_t + u^i \partial_i & \text{is tangent to } \bigcup_{0 \le t \le T} \{t\} \times \Omega(t), \quad (1d)\\
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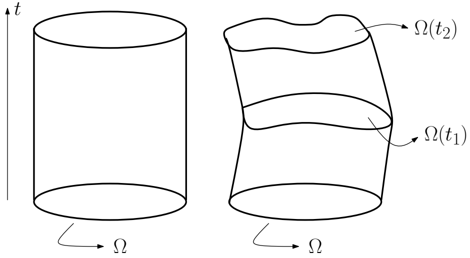

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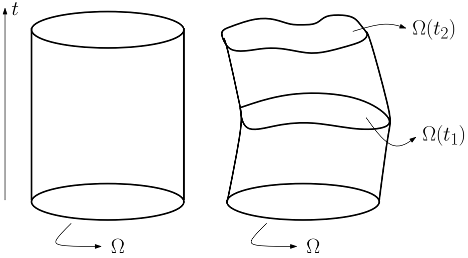

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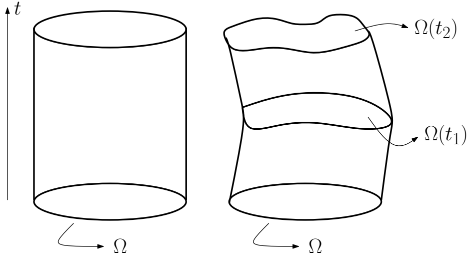

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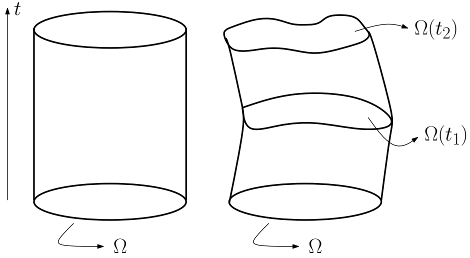

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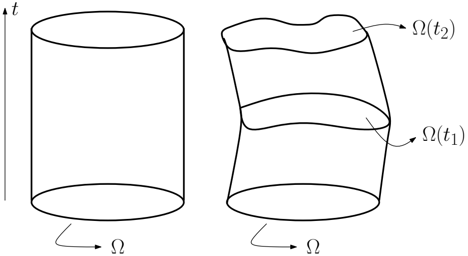

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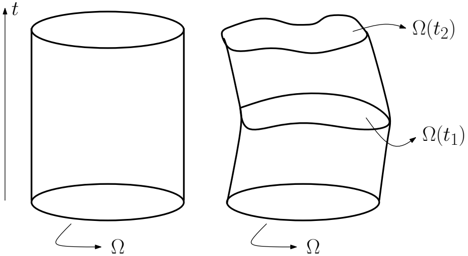

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The unknowns in (2) are η and p.

Theorem (D–, Ebin): Existence & uniqueness

Let $s > \frac{3}{2} + 2$, Ω be a bounded domain in $\mathbb{R}^3$ with a connected smooth boundary, and $u_0 \in H^s(\Omega, \mathbb{R}^3)$ be a divergence free vector field. Assume that $\kappa > 0$.

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$$\eta_\kappa \in C^0([0, T_\kappa), \mathcal{E}^s_\mu(\Omega)),\ \dot{\eta}_\kappa \in L^\infty([0, T_\kappa), H^s(\Omega)),$$
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$$p_\kappa \in L^\infty([0, T_\kappa), H^{s-\frac{1}{2}}(\Omega_\kappa(t))),$$
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where $\Omega_\kappa(t) = \eta_\kappa(t)(\Omega)$. (Solution in Eulerian coordinates, $(u_\kappa, p_\kappa, \Omega_\kappa(t))$, automatically follows.)

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More precisely, we would like to show that solutions to the free boundary Euler equations converge (in a suitable topology) to solutions of the fixed boundary Euler equations, when $\kappa \to \infty$.

The (fixed boundary) Euler equations

In order to state the next theorem, we need to introduce Euler's equations in the fixed domain Ω:

$$\begin{cases} \dfrac{\partial \vartheta}{\partial t} + (\vartheta \cdot \nabla)\vartheta = -\nabla \pi & \text{in } [0, T] \times \Omega, \quad (3a) \\ \operatorname{div}(\vartheta) = 0 & \text{in } \Omega, \quad (3b) \\ \langle \vartheta, \nu \rangle = 0 & \text{on } \partial\Omega, \quad (3c) \\ \vartheta(0) = \vartheta_0, & \quad (3d) \end{cases}$$

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The unknown in (3) is ϑ (π is completely determined by the velocity ϑ).

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Notice that $\mathcal{D}^s_\mu(\Omega) \subset \mathcal{E}^s_\mu(\Omega)$.

Theorem (D–, Ebin): Convergence

Let $s > \frac{3}{2} + 2$. Assume that Ω is a ball. Let $\{u_{0\kappa}\} \subset H^s(\Omega, \mathbb{R}^3)$ be a family of divergence free vector fields parametrized by the coefficient of surface tension κ, such that $u_{0\kappa}$ converges in $H^s(\Omega, \mathbb{R}^3)$, as $\kappa \to \infty$, to a divergence free vector field ϑ_0 which is tangent to the boundary.

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Then, if T is sufficiently small, we find that $T_\kappa \geq T$ for all κ sufficiently large, and, as $\kappa \to \infty$, $\eta_\kappa(t) \to \zeta(t)$ as a continuous curve in $\mathcal{E}^s_\mu(\Omega)$ (recall $\mathcal{D}^s_\mu(\Omega) \subset \mathcal{E}^s_\mu(\Omega)$).

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Convergence: summary

In a nutshell:

If the coefficient of surface tension κ goes to infinity, then solutions to the free-boundary Euler equations converge to solutions of the fixed boundary Euler equations.

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$$\int_0^t \nabla p_\kappa \circ \eta_\kappa \to \int_0^t \nabla \pi \circ \zeta,$$

in H^s for any $t > 0$.

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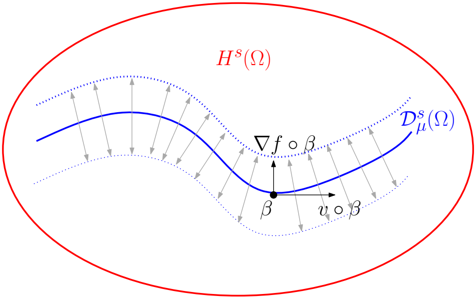
Convergence part of our theorem: D– and Ebin in 2d (2014).

Core of the proof: decomposition of η_κ



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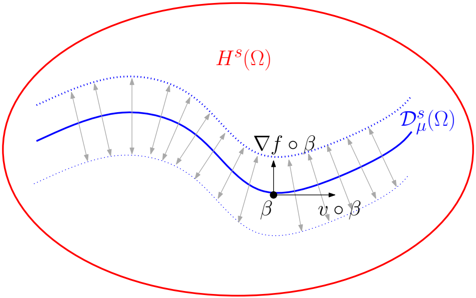

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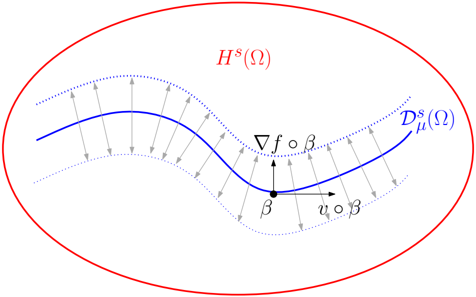

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The exponential map from the normal bundle to $H^s(\Omega, \mathbb{R}^3)$ is a diffeomorphism in a neighborhood of $\mathcal{D}^s_\mu(\Omega)$.

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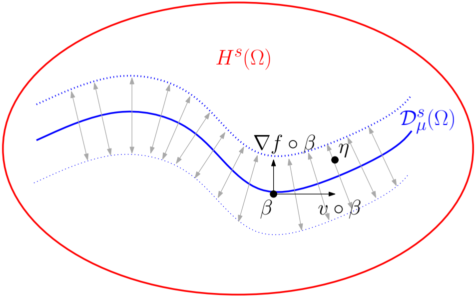

It follows that if η_κ is near $\mathcal{D}^s_\mu(\Omega)$, then there exist β_κ and ∇f_κ such that

$$\eta_\kappa = (\mathrm{id} + \nabla f_\kappa) \circ \beta_\kappa. \tag{5}$$

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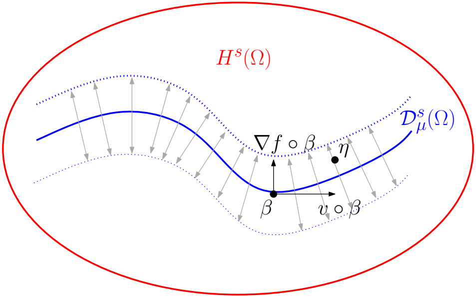

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Since $\eta_\kappa(0) = \mathrm{id} \in \mathcal{D}^s_\mu(\Omega)$, $\eta_\kappa(t)$ is near $\mathcal{D}^s_\mu(\Omega)$ for small time, and decomposition (5) applies.

Small oscillation

For the rest of the talk, assume: κ large, Ω a ball.

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decomposes η_κ as a motion that fixes the boundary β_κ:

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Goal: write the free boundary Euler equations as equations for f_κ and β_κ, and derive estimates showing that $\nabla f_\kappa \sim \frac{1}{\kappa}$, i.e., ∇f_κ is small.

Elliptic equation for f_κ

Since η_κ and β_κ are volume preserving, the Jacobian J gives

$$\begin{aligned} 1 = J(\eta_\kappa) &= J((\mathrm{id} + \nabla f_\kappa) \circ \beta_\kappa) \\ &= J(\mathrm{id} + \nabla f_\kappa)\, \underbrace{J(\beta_\kappa)}_{=1} \\ &= J(\mathrm{id} + \nabla f_\kappa). \end{aligned}$$

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Given $f_\kappa|_{\partial\Omega}$, equation (6) is a non-linear Dirichlet problem for f_κ. Therefore, if f_κ is small, it is determined by its boundary values.

Equation for $f_\kappa\big|_{\partial\Omega}$

Differentiating $\eta_\kappa = (\mathrm{id} + \nabla f_\kappa) \circ \beta_\kappa$ and using the original equation $\ddot{\eta}_\kappa = -\nabla p_\kappa \circ \eta_\kappa$, we obtain an equation for f_κ on the boundary:

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In (7) we have that $\partial_t \sim \partial_x^{\frac{3}{2}}$.

The boundary-interior system

$\mathscr{A}_\kappa(\beta_\kappa, v_\kappa, f_\kappa)$ depends on the interior values of f_κ ($\mathscr{A}_\kappa \sim \Delta_{\partial\Omega}\partial_\nu$), hence on the extension of f_κ to Ω, given by the previous elliptic equation. (Think of $\mathscr{A}_\kappa$ as Dirichlet-Neumann type of operator.)

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Solving the boundary-interior system; estimates

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is at the core of the work. The method is inspired on Kato's "The Cauchy problem for quasi-linear symmetric hyperbolic systems." The extra regularity $\nabla f_\kappa \in H^{s+\frac{3}{2}}(\Omega)$ gives that $\partial\Omega(t)$ is H^{s+1} regular.

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$$\| \nabla f_\kappa \|_{s+\frac{3}{2}} \leq \frac{C}{\kappa},\ \| \nabla \dot{f}_\kappa \|_s \leq \frac{C}{\sqrt{\kappa}}.$$

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(extra regularity of $\partial\Omega(t)$; recall previous comments on $\frac{3}{2}$ derivatives.) This essentially takes care of the convergence part of our result.

Hypothesis that Ω is a ball

Since

$$\eta_\kappa = (\mathrm{id} + \nabla f_\kappa) \circ \beta_\kappa = \beta_\kappa + \nabla f_\kappa \circ \beta_\kappa,$$

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Recalling that ∇p_κ enters in the equation and that $p_\kappa|_{\partial\Omega(t)} = \kappa\mathcal{A}$, $\mathcal{H}_\kappa$ gives a term like

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— Thank you for your attention —

Appendix: existence, determining the remaining quantities

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The function h_κ is harmonic and solves

$$\begin{cases} \Delta h_\kappa = 0, & \text{in } (\mathrm{id} + \nabla f_\kappa)(\Omega), \\ \frac{\partial h_\kappa}{\partial N_\kappa} = \langle (\nabla \dot{f}_\kappa + D_{v_\kappa} \nabla f_\kappa + v_\kappa) \circ (\mathrm{id} + \nabla f_\kappa)^{-1}, N_\kappa \rangle & \text{on } \partial(\mathrm{id} + \nabla f_\kappa)(\Omega). \end{cases}$$

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Finally, the pressure p_κ decomposes as $p_\kappa = p_{0,\kappa} + \kappa \mathcal{A}^H_\kappa$, where $p_{0,\kappa}$ solves

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($\mathcal{A}^H_\kappa$ has been taken care of in the f_κ equation).

Appendix, existence: solution via iteration

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Iterate the process, and obtain a fixed point.

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$$q = \lambda^2 p = \lambda^2 \kappa \mathcal{A} = \lambda^3 \kappa (1/\lambda)\mathcal{A},$$

so the scaled motion has an effective coefficient of surface tension of $\lambda^3\kappa$.