Generalized Thom spectra and topological Hochschild homology

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Outline

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Generalized Thom spectra

Let R be an E_∞ ring spectrum.

Then $R^*(X)$ is a multiplicative cohomology theory.

The space of units $\mathrm{GL}_1(R)$ represents the units in the ring $R^0(X)$:

$$R^0(X)^* \cong [X, \mathrm{GL}_1(R)]$$

Ando-Blumberg-Gepner-Hopkins-Rezk define an R-module Thom spectrum functor

$$M\colon \mathit{Top}/B\mathrm{GL}_1(R) \longrightarrow R\mathrm{-modules}$$

in the Elmendorf-Kriz-Mandell-May category of spectra.

For $R = \mathbb{S}$, the sphere spectrum, this gives the classical theory of Thom spectra over $BF \simeq B\mathrm{GL}_1(\mathbb{S})$.

We shall discuss how to implement a symmetric monoidal version of such a Thom spectrum functor in the setting of symmetric spectra. (A similar construction applies for orthogonal spectra).

The first step is to replace the category $\mathit{Top}/B\mathrm{GL}_1(R)$ by a (Quillen) equivalent symmetric monoidal category.

This requires that we find a strictly commutative model of $B\mathrm{GL}_1(R)$.

Background on $\mathcal{I}$-spaces and symmetric spectra

Let $\mathcal{I}$ be the category with objects the finite sets $\mathbf{n} = \{1, \ldots, n\}$ and morphisms the injective maps.

The ordered concatenation of ordered sets $\mathbf{m} \sqcup \mathbf{n}$ makes $\mathcal{I}$ a symmetric monoidal category.

Definition

The category of $\mathcal{I}$-spaces $Top^{\mathcal{I}}$ is the category of functors $X \colon \mathcal{I} \to Top$.

The $\sqcup$-product on $\mathcal{I}$ induces a symmetric monoidal "convolution product" $\boxtimes$ on $Top^{\mathcal{I}}$:

$$X \boxtimes Y(\mathbf{n}) = \operatorname*{colim}_{\mathbf{n}_1 \sqcup \mathbf{n}_2 \to \mathbf{n}} X(\mathbf{n}_1) \times Y(\mathbf{n}_2).$$

We use the term *$\mathcal{I}$-space monoid* for a monoid in $Top^{\mathcal{I}}$.

A map of $\mathcal{I}$-spaces $X \to Y$ is said to be an *$\mathcal{I}$-equivalence* if the map of homotopy colimits $X_{h\mathcal{I}} \to Y_{h\mathcal{I}}$ is a weak equivalence.

Theorem (Sagave-S.)

There is a symmetric monoidal Quillen equivalence

$$\mathrm{colim}\colon \mathit{Top}^{\mathcal{I}} \rightleftarrows \mathit{Top} : \mathrm{const}$$

and an induced Quillen equivalence

$$\{\textit{commutative } \mathcal{I}\textit{-space monoids}\} \simeq \{E_\infty \textit{ spaces}\}$$

The derived equivalence takes an $\mathcal{I}$-space X to the homotopy colimit $X_{h\mathcal{I}}$.

If M is a commutative $\mathcal{I}$-space monoid, then $M_{h\mathcal{I}}$ is an E_∞ space (with an action of the Barratt-Eccles operad).

Let Sp^Σ be the category of symmetric spectra with the symmetric monoidal structure given by the smash product $\wedge$.

There is a symmetric monoidal Quillen adjunction

$$\mathbb{S}^{\mathcal{I}} \colon Top^{\mathcal{I}} \rightleftarrows Sp^\Sigma : \Omega^{\mathcal{I}}$$

where $\mathbb{S}^{\mathcal{I}}[X]_n = S^n \wedge X(\mathbf{n})_+$ and $\Omega^{\mathcal{I}}(E)(\mathbf{n}) = \Omega^n(E_n)$.

If R is a (semistable) commutative symmetric ring spectrum, then $\Omega^{\mathcal{I}}(R)$ is a strictly commutative model of the E_∞ space $\Omega^\infty(R)$.

Definition

The *$\mathcal{I}$-space units* $\mathrm{GL}_1^{\mathcal{I}}(R)$ of R is the sub commutative $\mathcal{I}$-space monoid of $\Omega^{\mathcal{I}}(R)$ such that $\mathrm{GL}_1^{\mathcal{I}}(R)(\mathbf{n})$ is the union of the path components in $\Omega^n(R_n)$ that represent units in the commutative ring $\pi_0(R) = \operatorname{colim}_n \pi_n(R_n)$.

Remark

There is a map of commutative symmetric ring spectra

$$\mathbb{S}^{\mathcal{I}}[\mathrm{GL}_1^{\mathcal{I}}(R)] \to R$$

analogous to the algebraic situation where a commutative ring receives a homomorphism from the integral group ring of its units.

Notation

We write G for $\mathrm{GL}_1^{\mathcal{I}}(R)$ (or for a cofibrant replacement)

The classifying space BG can be defined by a bar construction in $Top^{\mathcal{I}}$: $BG = B^{\boxtimes}(*, G, *)$. This is a commutative $\mathcal{I}$-space monoid.

Definition

The universal G-fibration $EG \twoheadrightarrow BG$ is defined by a factorization $B^{\boxtimes}(*, G, G) \overset{\sim}{\rightarrowtail} EG \twoheadrightarrow BG$ in the category of commutative $\mathcal{I}$-space monoids.

Let $Top^{\mathcal{I}}_G$ be the category of G-modules in $Top^{\mathcal{I}}$.

Proposition

There are symmetric monoidal Quillen equivalences

$$Top^{\mathcal{I}}/BG \underset{U}{\overset{\longleftarrow}{\longrightarrow}} Top^{\mathcal{I}}_G/EG \overset{\longrightarrow}{\longleftarrow} Top^{\mathcal{I}}_G$$

where $U(X \xrightarrow{\alpha} BG)$ is given by the pullback

$$\begin{array}{ccc} U(\alpha) & \longrightarrow & EG \\ \downarrow & & \downarrow \\ X & \xrightarrow{\alpha} & BG. \end{array}$$

This justifies the term "classifying space" for BG.

The R-module Thom spectrum functor on $Top^{\mathcal{I}}/BG$

Let R be a (flat) commutative symmetric ring spectrum, and let Sp^{Σ}_{R} be the category of R-modules in Sp^{Σ}.

Definition

The R-module Thom spectrum functor $T^{\mathcal{I}}$ is given by

$$T^{\mathcal{I}}\colon Top^{\mathcal{I}}/BG \xrightarrow{U} Top^{\mathcal{I}}_{G}/EG \xrightarrow{\mathbb{S}^{\mathcal{I}}} Sp^{\Sigma}_{\mathbb{S}^{\mathcal{I}}[G]}/\mathbb{S}^{\mathcal{I}}[EG] \xrightarrow{B(-,\mathbb{S}^{\mathcal{I}}[G],R)} Sp^{\Sigma}_{R}/\mathrm{MGL}^{\mathcal{I}}_{1}(R)$$

where

- the two-sidet bar construction $B(\mathbb{S}^{\mathcal{I}}[U(\alpha)],\mathbb{S}^{\mathcal{I}}[G],R)$ is a homotopy invariant version of $\mathbb{S}^{\mathcal{I}}[U(\alpha)]\wedge_{\mathbb{S}^{\mathcal{I}}[G]} R$,
- $\mathrm{MGL}^{\mathcal{I}}_{1}(R) = B(\mathbb{S}^{\mathcal{I}}[EG],\mathbb{S}^{\mathcal{I}}[G],R)$.

The symmetric monoidal product on $Top^{\mathcal{I}}/BG$ takes a pair of objects $\alpha\colon X \to BG$ and $\beta\colon Y \to BG$ to

$$\alpha \boxtimes \beta\colon X \boxtimes Y \to BG \boxtimes BG \to BG$$

where the last map is the multiplication in BG.

Theorem

The Thom spectrum functor $T^{\mathcal{I}}$ is lax symmetric monoidal and the derived monoidal structure maps

$$T^{\mathcal{I}}(\alpha)^{\mathrm{cof}} \wedge_R T^{\mathcal{I}}(\beta)^{\mathrm{cof}} \to T^{\mathcal{I}}(\alpha) \wedge_R T^{\mathcal{I}}(\beta) \to T^{\mathcal{I}}(\alpha \boxtimes \beta)$$

are stable equivalences.

Generalized Thom spectra from space level data

The homotopy colimit $BG_{h\mathcal{I}}$ is an E_∞ model of the classifying space $B\mathrm{GL}_1(R)$ and there is a lax monoidal Quillen equivalence

$$\text{“}\mathcal{I}\text{-spacification”}: \mathit{Top}/BG_{h\mathcal{I}} \xrightarrow{\simeq} \mathit{Top}^{\mathcal{I}}/BG$$

(not symmetric monoidal).

Definition

The Thom spectrum functor T on $\mathit{Top}/BG_{h\mathcal{I}}$ is the composition

$$T\colon \mathit{Top}/BG_{h\mathcal{I}} \xrightarrow{\simeq} \mathit{Top}^{\mathcal{I}}/BG \xrightarrow{T^{\mathcal{I}}} \mathit{Sp}^{\Sigma}_{R}/M\mathrm{GL}_1^{\mathcal{I}}(R).$$

Theorem

- *The Thom spectrum functor T is lax monoidal and takes weak homotopy equivalences over $BG_{h\mathcal{I}}$ to stable equivalences.*
- *If $\mathcal{C}$ is an operad augmented over the Barratt-Eccles operad, then there is an induced homotopy functor*

$$T\colon Top[\mathcal{C}]/BG_{h\mathcal{I}} \to Sp_R^{\mathcal{I}}[\mathcal{C}]/\mathrm{MGL}_1^{\mathcal{I}}(R)$$

between the corresponding categories of $\mathcal{C}$-algebras.

Thom spectra associated to $SU(n)$

We consider R-algebra Thom spectra $T(SU(n))$ associated to loop maps $SU(n) \to BG_{h\mathcal{I}}$ and we analyze the filtration by $T(SU(m))$ for $m \leq n$.

Proposition

For $m < n$ there are homotopy pushout squares

$$\begin{array}{ccc} T(\Sigma\mathbb{C}P^{m-1})^{\mathrm{cof}} \wedge_R T(SU(m)) & \longrightarrow & T(SU(m)) \\ \downarrow & & \downarrow \\ T(\Sigma\mathbb{C}P^{m})^{\mathrm{cof}} \wedge_R T(SU(m)) & \longrightarrow & T(SU(m+1)). \end{array}$$

Proof.

There are embeddings $\Sigma\mathbb{C}P^{m-1} \to SU(m)$ such that the outer diagrams

$$\begin{array}{ccccc} \Sigma\mathbb{C}P^{m-1} \times SU(m) & \longrightarrow & SU(m) \times SU(m) & \longrightarrow & SU(m) \\ \downarrow & & \downarrow & & \downarrow \\ \Sigma\mathbb{C}P^{m} \times SU(m) & \longrightarrow & SU(m+1) \times SU(m) & \longrightarrow & SU(m+1) \end{array}$$

are pushout diagrams. Now apply the Thom spectrum functor T □

We must analyze the R-modules $T(\Sigma\mathbb{C}P^m)$.

Lemma

Let $\Sigma X \to BG$ be a map of based $\mathcal{I}$-spaces with adjoint $\alpha\colon X \to \Omega(BG) \simeq \mathrm{GL}_1(R)$. Then there is a homotopy pushout square

$$\begin{array}{ccc} \mathbb{S}^{\mathcal{I}}[X] \wedge R & \xrightarrow{\text{proj}} & R \\ \downarrow{\scriptstyle\alpha} & & \downarrow \\ R & \longrightarrow & T^{\mathcal{I}}(\Sigma X). \end{array}$$

This gives a homotopy cofiber sequence

$$R \wedge X_{h\mathcal{I}} \to R \to T^{\mathcal{I}}(\Sigma X).$$

Applies in particular to $\Sigma\mathbb{C}P^1 = SU(2) \to BG_{h\mathcal{I}}$.

Now suppose that $\pi_*(R)$ is concentrated in even degrees.

Then $R^*(\mathbb{C}P^m) = \pi_*(R)[x]/x^{m+1}$, for $x \in R^2(\mathbb{C}P^m)$.

The composition $\Sigma\mathbb{C}P^{n-1} \to SU(n) \to BG_{h\mathcal{I}}$ has adjoint

$$u\colon \mathbb{C}P^{n-1} \to \Omega(BG_{h\mathcal{I}}) \simeq G_{h\mathcal{I}}.$$

Let $u_i \in \pi_{2i}(R)$ for $i = 1, \ldots, n-1$ be such that

$$[u] = 1 + u_1 x + u_2 x^2 + \cdots + u_{n-1} x^{n-1} \in R^0(\mathbb{C}P^{n-1})^*$$

The splitting $R \wedge \mathbb{C}P^{n-1} \simeq \bigvee_{i=1}^{n-1} \Sigma^{2i} R$ gives homotopy cofiber sequences

$$\Sigma^{2m} R \to T(\Sigma\mathbb{C}P^{m-1}) \to T(\Sigma\mathbb{C}P^m)$$

Applying $(-) \wedge_R T(SU(m))^{\mathrm{cof}}$, the previous results imply:

Proposition

There are homotopy cofiber sequences

$$\Sigma^{2m} T(SU(m)) \xrightarrow{u_m} T(SU(m)) \to T(SU(m+1)).$$

Regular quotients as Thom spectra

Suppose again that $\pi_*(R)$ is concentrated in even degrees.

Given elements $u_i \in \pi_{2i}(R)$ for $i = 1, \ldots, n-1$, let

$$u = 1 + u_1 x + u_2 x^2 + \cdots + u_{n-1} x^{n-1} \in R^0(\mathbb{C}P^{n-1})^*$$

be represented by a map $u \colon \mathbb{C}P^{n-1} \to G_{h\mathcal{I}} \simeq \Omega(BG_{h\mathcal{I}})$.

Theorem

The adjoint $\Sigma\mathbb{C}P^{n-1} \to BG_{h\mathcal{I}}$ can be extended to a loop map $SU(n) \to BG_{h\mathcal{I}}$, and if $u_1, \ldots, u_{n-1}$ is a regular sequence in $\pi_*(R)$, then the R-algebra $T(SU(n))$ is a regular quotient of R:

$$T(SU(n)) \simeq R/(u_1, \ldots, u_{n-1})$$

Remark

The theorem shows that for every choice of elements $u_i \in \pi_{2i}(R)$ for $i = 1, \ldots, n-1$, there exists a sequence of R-algebras

$$R = T(1) \to T(2) \to \cdots \to T(n-1) \to T(n)$$

such that there are cofibration sequences

$$\Sigma^{2m} T(m) \xrightarrow{u_m} T(m) \to T(m+1).$$

(Take $T(m) = T(SU(m))$).

Topological Hochschild homology

Let R be a commutative symmetric ring spectra, and let A be a (not necessarily commutative) R-algebra.

The cyclic bar construction $B_R^{\mathrm{cy}}(A)$ is the realization of the simplicial R-module

$$[k] \mapsto \underbrace{A \wedge_R A \wedge \cdots \wedge_R A}_{k+1}$$

If A is cofibrant, then $B_R^{\mathrm{cy}}(A)$ is a model of the topological Hochschild homology $\mathrm{THH}^R(A)$.

For a general R-algebra A, we define $\mathrm{THH}^R(A) = B_R^{\mathrm{cy}}(A^{\mathrm{cof}})$, where A^{cof} is a cofibrant replacement of A.

Topological Hochschild homology of Thom spectra

Let M be an $\mathcal{I}$-space monoid, and let $\alpha\colon M \to BG$ be a map of $\mathcal{I}$-space monoids.

Then the Thom spectrum $T^{\mathcal{I}}(\alpha)$ is an R-algebra.

Theorem

There is a stable equivalence of R-modules

$$\mathrm{THH}^R(T^{\mathcal{I}}(\alpha)) \simeq T^{\mathcal{I}}\big(B^{cy}(M) \xrightarrow{B^{cy}(\alpha)} B^{cy}(BG) \to BG\big),$$

where $B^{cy}(BG) \to BG$ is the iterated multiplication in BG.

Proof.

We have

$$\underbrace{T^{\mathcal{I}}(\alpha)^{\mathrm{cof}} \wedge_R \cdots \wedge_R T^{\mathcal{I}}(\alpha)^{\mathrm{cof}}}_{k+1} \xrightarrow{\simeq} T^{\mathcal{I}}\big(M^{\boxtimes(k+1)} \to BG^{\boxtimes(k+1)} \to BG\big)$$

for each $k \geq 0$. $\square$

Reformulation in terms of loop spaces

Let $f\colon X \to BG_{h\mathcal{I}}$ be a loop map, $f \simeq \Omega(Bf)$, for a based map $Bf\colon BX \to B^2BG_{h\mathcal{I}}$. Then $T(f)$ is an R-algebra.

Let $L(BX)$ be the free loop space and

$$L^\eta(Bf)\colon L(BX) \xrightarrow{L(Bf)} L(B^2G_{h\mathcal{I}}) \simeq B^2G_{h\mathcal{I}} \times BG_{h\mathcal{I}} \xrightarrow{\{\eta,\mathrm{id}\}} BG_{h\mathcal{I}}$$

where η is induced by the Hopf map.

Theorem

- *If f is a loop map, then*
$$\mathrm{THH}^R(T(f)) \simeq T(L^\eta(Bf)).$$
- *If f is a 2-fold loop map, then*
$$\mathrm{THH}^R(T(f)) \simeq T(f) \wedge_R T(\eta \circ Bf)^{\mathrm{cof}}.$$
- *If f is a 3-fold loop map, then*
$$\mathrm{THH}^R(T(f)) \simeq T(f) \wedge BX_+.$$

Example (Work in progress)

Let E_n be the 2-periodic Lubin-Tate spectrum,

$$\pi_*(E_n) = W(\mathbb{F}_{p^n})[[u_1, \dots, u_{n-1}]][u, u^{-1}], \quad |u_i| = 0, \ |u| = 2$$

The 2-periodic Morava K-theory spectrum K_n is given by

$$K_n = E_n/(p, u_1, \dots, u_{n-1}), \quad \pi_*(K_n) = \mathbb{F}_{p^n}[u, u^{-1}].$$

Thus, there exists a loop map $f\colon SU(n+1) \to \mathrm{BGL}_1(E_n)$ such that $T(f) \simeq K_n$ as an E_n-algebra.

The algebra structure on $T(f) \simeq K_n$ depends on the map $f\colon SU(n+1) \to \mathrm{BGL}_1(E_n)$. Using this we prove:

Theorem

For each $k \geq 1$ such that $p \geq (n+1)(k+1)+1$, there exists an E_n-algebra structure on K_n for which

$$\mathrm{THH}_*^{E_n}(K_n) \cong \bigoplus_{i=1}^{k} \pi_*(E_n)/(p, u_1, \ldots, u_{n-1})^{\infty}$$

Here $\pi_*(E_n)/(p, u_1, \ldots, u_{n-1})^{\infty}$ denotes the $\pi_*(E_n)$-module

$$\operatorname*{colim}_{i, j_1, \ldots, j_{n-1}} \pi_*(E_n)/(p^i, u_1^{j_1}, \ldots, u_{n-1}^{j-1}).$$

This complements work of Vigleik Angeltveit.

Graded Thom spectra

For $R = \mathbb{S}$, composing with the maps $BO \to BF \xrightarrow{\simeq} \mathrm{GL}_1(\mathbb{S})$, we get the classical Thom spectra for stable vector bundles $X \to BO$.

One may also consider graded Thom spectra associated to virtual vector bundles $X \to BO \times \mathbb{Z}$.

For instance, we have the periodic cobordism spectra

$$MOP \simeq \bigvee_{n\in\mathbb{Z}} \Sigma^n MO, \quad MUP \simeq \bigvee_{n\in\mathbb{Z}} \Sigma^{2n} MU$$

and the connective versions

$$MOP_{\geq 0} \simeq \bigvee_{n\geq 0} \Sigma^n MO, \quad MUP_{\geq 0} \simeq \bigvee_{n\geq 0} \Sigma^{2n} MU$$

In the periodic cases, it is natural to consider a logarithmic version of topological Hochschild homology.

Pre-log ring spectra

In algebraic geometry, a pre-log ring (A, M) is given by

- a commutative ring A,
- a commutative monoid M,
- a monoid homomorphism $M \to (A, \cdot)$.

The localization $A \to A[M^{-1}]$ admits a factorisation

$$(A, \{1\}) \to (A, M) \to (A[M^{-1}], M^{gp})$$

in the category of pre-log rings.

This was used by Hesselholt-Madsen in their work on algebraic K-theory of local fields.

In joint work with Rognes-Sagave, we have introduced a analogous notion of a pre-log ring spectrum (A, M) for a commutative symmetric ring spectrum A.

Logarithmic topological Hochschild homology

There is a logarithmic version of topological Hochschild homology $\mathsf{THH}(A, M)$ that is sometimes better behaved than $\mathsf{THH}(A[M^{-1}])$.

In particular, certain types of pre-log ring spectra (A, M) gives rise to THH-localization sequences.

Theorem (Rognes-Sagave-S)

Let E be a d-periodic commutative symmetric ring spectrum with connective cover $j\colon e \to E$. Then there is a homotopy cofiber sequence

$$\mathsf{THH}(e[0, d\rangle) \to \mathsf{THH}(e) \to \mathsf{THH}(e, j_*\mathrm{GL}_1^{\mathcal{J}}(E))$$

where $(e, j_*\mathrm{GL}_1^{\mathcal{J}}(E))$ is the pre-log ring spectrum obtained by pulling back the graded units $\mathrm{GL}_1^{\mathcal{J}}(E)$ of E.

In some cases, such as $e = ku$, the algebra structure of $\mathsf{THH}(e, j_*\mathrm{GL}_1^{\mathcal{J}}(E))$ is more regular than that of $\mathsf{THH}(e)$.

Logarithmic topological Hochschild homology of $MUP_{\geq 0}$

There is a canonical pre-log ring spectrum $(MUP_{\geq 0}, V)$ such that $MUP_{\geq 0}[V^{-1}] \simeq MUP$.

Theorem

There is a homotopy cofiber sequence

$$\mathrm{THH}(MU) \to \mathrm{THH}(MUP_{\geq 0}) \to \mathrm{THH}(MUP_{\geq 0}, V)$$

where

- $\mathrm{THH}(MU) \simeq MU \wedge SU_+$
- $\mathrm{THH}(MUP_{\geq 0}) = MU \wedge SU_+ \vee MUP_{>0} \wedge U_+$
- $\mathrm{THH}(MUP_{\geq 0}, V) \simeq MUP_{\geq 0} \wedge U_+$