# High-dimensional permutations and discrepancy

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A line here means either a row or a column.

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Whereas a matrix has two kinds of lines, namely rows and columns, now there are $d+1$ kinds of lines.
A line is a set of $n$ entries in the array that are obtained by fixing $d$ out of the $d+1$ coordinates and the letting the remaining coordinate take all values from 1 to $n$.

# The case $d = 2$. A familiar face?

According to our definition, a 2-dimensional permutation on $[n]$ is an $[n] \times [n] \times [n]$ array of zeros and ones in which every row, every column, and every shaft contains exactly one 1-entry.

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According to our definition, a 2-dimensional permutation on $[n]$ is an $[n] \times [n] \times [n]$ array of zeros and ones in which every row, every column, and every shaft contains exactly one 1-entry.
An equivalent description can be achieved by using a topographical map of this terrain.

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It is easily verified that the defining condition is that in this array every row and every column contains every entry $n \geq i \geq 1$ exactly once.

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It is easily verified that the defining condition is that in this array every row and every column contains every entry $n \geq i \geq 1$ exactly once.
In other words: Two-dimensional permutations are synonymous with Latin Squares.

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We seek high-dimensional counterparts of known phenomena in ("classical" = "one-dimensional") permutations. Specifically, we wish to:

- Enummerate $d$-dimensional permutations.
- Find how to generate them randomly and efficiently and describe their typical behavior.
- Investigate analogs of the Birkhoff von-Neumann Theorem on doubly stochastic matrices.

# … and more and more and more….

- Of Erdős-Szekeres.

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- Of Erdős-Szekeres. Of the solution to Ulam's Problem.
- Find out how small their discrepancy can be.
- Use low-discrepancy permutations to construct high-dimensional expanders.

# Erdős-Szekeres for high-dimensional permutations, and a word on Ulam's problem

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*Every d-dimensional permutation has a monotone subsequence of length $\Omega_d(\sqrt{n})$. The bound is tight up to the implicit coefficient.*

# Erdős-Szekeres for high-dimensional permutations, and a word on Ulam's problem

## Theorem (NL+Michael Simkin)

*Every d-dimensional permutation has a monotone subsequence of length $\Omega_d(\sqrt{n})$. The bound is tight up to the implicit coefficient.*
*In almost every d-dimensional permutation the length of the longest monotone subsequence is $\Theta_d(n^{\frac{d}{d+1}})$.*

# The count - An interesting numerology

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$$n! = \left((1 + o(1))\frac{n}{e}\right)^n$$

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As van Lint and Wilson showed, the number of order-$n$ Latin squares is

$$|\mathcal{L}_n| = \left((1+o(1))\frac{n}{e^2}\right)^{n^2}$$

# So, let us conecture

## Conjecture

*The number of d-dimensional permutations on [n] is*

$$|S_n^d| = \left((1+o(1))\frac{n}{e^d}\right)^{n^d}$$

# and what we actually know

At present we can only prove the upper bound

**Theorem (NL, Zur Luria '14)**

*The number of d-dimensional permutations on [n] is*

$$|S_n^d| \leq \left((1 + o(1))\frac{n}{e^d}\right)^{n^d}$$

# How van Lint and Wilson enumerated Latin Squares

Recall that the permanent of a square matrix is a "determinant without signs".

$$per(A) = \sum_{\sigma \in S_n} \prod a_{i,\sigma(i)}$$

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- It counts perfect matchings in bipartite graphs.
- In other words, it counts the generalized diagonals included in a $0/1$ matrix.
- It is $\#$-$P$-hard to calculate the permanent exactly, even for a $0/1$ matrix.
- On the other hand, there is an efficient approximation scheme for permanents of nonnegative matrices.
- The most important open problem in algebraic computational complexity is to separate permanents from determinants.

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What is min $per\ A$ over $n \times n$ doubly-stochastic matrices?

What is min *per* $A$ over $n \times n$ doubly-stochastic matrices? As conjectured by van der Waerden in the 20's and proved over 50 years later, in the minimizing matrix all entries are $\frac{1}{n}$.

## Theorem (Falikman; Egorichev '80-81)

*The permanent of every $n \times n$ doubly stochastic matrix is $\geq \frac{n!}{n^n}$.*

# An upper bound on permanents

The following was conjectured by Minc

## Theorem (Brégman '73)

*Let $A$ be an $n \times n$ $0/1$ matrix with $r_i$ ones in the $i$-th row $i = 1, \ldots, n$. Then per $A \leq \prod_i (r_i!)^{1/r_i}$. The bound is tight.*

# How we proved the upper bound on the number of $d$-dimensional permutations

Our proof can be viewed as an extension of the Minc-Brégman theorem. Specifically, we use ideas from papers of Schrijver and Radhakrishnan elaborating on Brégman's proof.

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What about a matching lower bound?
The analog of the van der Waerden conjecture fails in higher dimension

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It is conceivable that an appropriate adaptation of his method will prove the tight lower bound in all dimensions.

# Approximately counting Latin squares

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The general scheme: We consider a Latin square (= a 2-dimensional permutation) $A$, layer by layer.
Namely, $A$ is an $n \times n \times n$ array of $0/1$ where every line has a single 1 entry.
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Note that every layer in $A$ is a permutation matrix.
Given several layers in $A$, how many permutation matrices can play the role of the next layer?

# How many choices for the next layer?

Let $B$ be a $0/1$ matrix where $b_{ij} = 1$ iff in all previous layers the $ij$ entry is zero.

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The set of all possible next layers coincides with the collection of generalized diagonals in $B$. Therefore, there are exactly $\text{per}B$ possibilities for the next layer.

# How many choices for the next layer?

To estimate the number of Latin squares we bound at each step the number of possibilities for the next layer ($=\text{per}B$) from above and from below using Minc-Brégman and van der Waerden, respectively.

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Repeat....

# Our high-dim Minc-Brégman Theorem

## Definition

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## Theorem

$$per_d(A) \leq \prod_{\mathbf{i}} \exp(f(d, r_{\mathbf{i}})),$$

*where $r_{\mathbf{i}}$ is the number of $1$'s in the line $l_{\mathbf{i}}$. (All lines in some specific direction).*

# Our high-dimensional Minc-Brégman Theorem (contd.)

We define $f(d,r)$ via $f(0,r) = \log r$, and

$$f(d,r) = \frac{1}{r} \sum_{k=1,\dots,r} f(d-1,k).$$

Note that $f(1,r) = \frac{\log(r!)}{r}$ and we recover Brégman's inequality.

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Note that $f(1, r) = \frac{\log(r!)}{r}$ and we recover Brégman's inequality. In general

$$f(d, r) = \log r - d + O_d(\frac{\log^d r}{r})$$

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High-dimensional permutations have something interesting to offer here. If yo want to learn about Discrepancy, the best place to go in Jirka's beautiful book.

# A little background - An example of discrepancy in geometry

Theorem (van Aardenne-Ehrenfest '45, Schmidt '75)

- *There is a set of* $N$ *points* $X \subset [0,1]^2$, *s.t.* $||X \cap R| - N \cdot area(R)| \leq O(\log N)$ *for every axis-parallel rectangle* $R \subseteq [0,1]^2$.

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- *On the other hand, for every set of N points* $X \subset [0,1]^2$ *there is an axis-parallel rectangle R for which* $||X \cap R| - N \cdot area(R)| \geq \Omega(\log N)$.

# Discrepancy in graph theory

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*Let $G = (V, E)$ be an n-vertex d-regular graph, and let $\lambda$ be the largest absolute value of a nontrivial eigenvalue of G's adjacency matrix. Then for every $A, B \subset V$,*

$$\left| e(A, B) - \frac{d}{n}|A||B| \right| \leq \lambda\sqrt{|A||B|}.$$

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*There exist order-$N$ Latin squares such that for every $A, B, C \subseteq [N]$ there holds*

$$\left| |L \cap (A \times B \times C)| - \frac{|A||B||C|}{N} \right| \leq O(\sqrt{|A||B||C|}).$$

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$$\left| |L \cap (A \times B \times C)| - \frac{|A||B||C|}{N} \right| \leq O(\sqrt{|A||B||C|}).$$

*Moreover, this holds for almost every Latin square.*

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## Note

*Every Latin square has an empty box of volume $\Omega(N^2)$.*

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## Theorem (Kedlaya '95)

*The Latin square of every order-$N$ group contains an empty box of volume $\geq \Omega(N^{2.357\cdots})$ (this exponent is $\frac{33}{14}$).*

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# A word about the proof

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We construct Latin squares with no large empty boxes using Keevash's construction of Steiner systems.

- To every Steiner triple system $X$ we associate a Latin square $L$ where $\{i, j, k\} \in X$ implies $L(i, j, k) = \ldots = L(k, j, i) = 1$ (six terms). Also, for all $i$, let $L(i, i, i) = 1$.

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- Keevash's method starts with a random greedy choice of triples. His main argument shows that whp this triple system can be completed to an STS. We show that this initial phase suffices to hit all boxes of volume above $Cn^2$.

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where $X$ is the $n \times n \times n$ array whose entries are zero in $A \times B \times C$ and one otherwise. Our Brégman-type upper bound on $\mathrm{per}_d X$ yields the conclusion fairly straightforwardly.

# Late breaking news

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Wonderful! But now we want more....

# Refining our questions on small discrepancy

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A one factorization of the complete graph $K_n$ (for even $n$) can be viewed as a Latin square that is symmetric and has $n$'s on the main diagonal. We can now ask questions such as:

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## Open Problem

*Do there exist one-factorizations in which the union of any $d$ (perhaps even $d = d(n)$?) color classes is a Ramanujan graph?*

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- Settle the strong discrepancy and sparsity problems.
- Find explicit constructions of low-discrepancy high-dimensional permutations.
- Find how to sample high-dimensional permutations and determine their typical behavior.