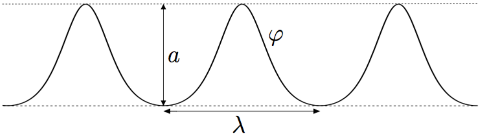
On periodic traveling waves of the Camassa-Holm equation

Anna Geyer

joint work with Jordi Villadelprat



Theoretical and Computational Aspects
of Nonlinear Surface Waves

BIRS
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The **Camassa-Holm equation**

$$u_t - u_{txx} + 2ku_x + 3\,u\,u_x = 2\,u_x u_{xx} + u\,u_{xxx}, \qquad \text{(CH)}$$

- arises as a shallow water approximation of the Euler equations for inviscid incompressible homogeneous fluids.
- $u = u(t, x)$ represents the water's free surface and $k \in \mathbb{R}$ is a parameter related to the critical shallow water speed.
- The CH equation is completely integrable and
- models wave breaking.

R. Camassa and D. Holm, An integrable shallow water equation with peaked solitons, Phys. Rev. Lett. 71 (1993)

Camassa-Holm equation

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Traveling Wave Solutions: $u(x,t) = \varphi(x - c\,t)$

$$\varphi''(\varphi - c) + \frac{(\varphi')^2}{2} + r + (c - 2k)\,\varphi - \frac{3}{2}\varphi^2 = 0 \qquad (1)$$

where c is the wave speed and $r \in \mathbb{R}$ is an integration constant.

J. Lenells, Traveling wave solutions of the Camassa-Holm equation, JDE '05.

Camassa-Holm equation

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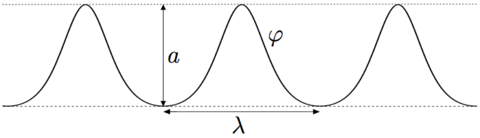
We will concentrate on **smooth periodic TWS.**



λ ... wave length $\quad a$ ... wave height

Proposition (Waves $\longleftrightarrow$ Orbits)

- φ is a smooth periodic solution of (2)

$$\varphi''(\varphi - c) + \frac{(\varphi')^2}{2} + r + (c - 2k)\,\varphi - \frac{3}{2}\varphi^2 = 0,$$

if and only if $(w, v) = (\varphi - c, \varphi')$ is a periodic orbit of the planar system

$$\begin{cases} w' = v, \\ v' = -\dfrac{A'(w) + \frac{1}{2}\,v^2}{w}, \end{cases} \tag{3}$$

where

$$A(w) := \alpha w + \beta w^2 - \frac{1}{2} w^3,$$

with $\alpha := r - 2kc - \frac{1}{2}c^2$ and $\beta := -(c + k)$.

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$$H(w, v) := \frac{wv^2}{2} + A(w).$$

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- Every periodic orbit of (3) belongs to the period annulus $\mathscr{P}$ of a center, which exists if and only if $-2\beta^2 < 3\alpha < 0$.

Observations:

periodic solution $\boldsymbol{\varphi}$	$\longleftrightarrow$	periodic orbit $\boldsymbol{\gamma}$
wave lenght $\boldsymbol{\lambda}$ of φ	$=$	period $\boldsymbol{T}$ of γ
wave height $\boldsymbol{a}$ of φ	$=$	$\boldsymbol{\ell(h)}$, where ℓ is an analytic diffeo with $\ell(h_0) = 0$.
$\{\varphi_a\}_{a\in(0,a_M)}$	$\longleftrightarrow$	$\{\gamma_h\}_{h\in(h_0,h_1)}$

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Deduce qualitative properties of the function λ from those of the **period function T**.

Result: $\lambda(a)$ is either unimodal or monotonous.

Theorem (A.G. & J. Villadelprat)

Given c, k, $c \neq -k$, there exist real numbers $r_1 < r_{b_1} < r_{b_2} < r_2$ such that the ***Camassa-Holm equation***

$$u_t + 2k\,u_x - u_{txx} + 3\,u\,u_x = 2\,u_x u_{xx} + u\,u_{xxx} \qquad \text{(CH)}$$

has ***smooth periodic TWS*** $\varphi(x - c\,t)$ *satisfying*

$$\varphi''(\varphi - c) + \frac{(\varphi')^2}{2} + r + (c - 2k)\,\varphi - \frac{3}{2}\varphi^2 = 0,$$

if and only if the integration constant $r \in (r_1, r_2)$.

The set of smooth periodic TWS form a continuous family $\{\varphi_a\}_a$ parametrized by the wave height a.

The ***wave length*** $\lambda = \lambda(a)$ *of φ_a satisfies the following:*

- *If $r \in (r_1, r_{b_1}]$, then $\lambda(a)$ is monotonous increasing.*
- *If $r \in (r_{b_1}, r_{b_2})$, then $\lambda(a)$ has a unique critical point (maximum).*
- *If $r \in [r_{b_2}, r_2)$, then $\lambda(a)$ is monotonous decreasing.*

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The bifurcation values are $r_1 = -\frac{2}{3}(k - \frac{1}{2}c)^2$, $r_2 = c(\frac{1}{2}c + 2k)$, and $r_{b_1} = k(c - \frac{1}{2}k)$, $r_{b_2} = \frac{\sqrt{6}-3}{12}(3k\sqrt{6} + 2c + 8k)(k\sqrt{6} - 2c - 2k)$

- For the *Degasperis-Procesi equation*

$$u_t + 2ku_x + 4uu_x - u_{txx} = 3u_x u_{xx} + uu_{xxx},$$

  the TWS satisfy an equation of the form

$$\varphi''(\varphi - c) + (\varphi')^2 + r + (c - 2k)\,\varphi - 2\varphi^2 = 0,$$

  and the wave length is qualitatively the same as for CH.

The wave length $\lambda(a)$ is monotonous for periodic TWS of

- Korteweg-de Vries and BBM equation:

$$u_t + u_x + \frac{3}{2}uu_x + \alpha u_{xxx} + \beta u_{txx} = 0,$$

with $\alpha, \beta \in \mathbb{R}$, whose TWS satisfy an equation of the form

$$\varphi'' + f(\varphi) = 0,$$

where f is quadratic.
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- generalized reduced Ostrovsky:

$$(u_t + u^p u_x)_x = u,$$

with $p \in \mathbb{N}$, whose TWS satisfy an equation of the form

$$\varphi''(\varphi^p - c) + p\,\varphi^{(p-1)}(\varphi')^2 - \varphi = 0.$$

$\rightsquigarrow$ classical monotonicity criteria do not apply.

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- potential systems:

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- systems with quadratic-like centers:

$$H(x, y) = A(x) + B(x)y + C(x)y^2$$

  [A. Garijo, J. Villadelprat, JDE '14]

Consider a Hamiltonian differential system of the form

$$\begin{cases} x' = y, \\ y' = -V'(x), \end{cases} \qquad H(x,y) = \frac{y^2}{2} + V(x),$$

where $V(x)$ is a quadratic potential. For the **period function**

$$T(h) = \int_{\gamma_h} \frac{dx}{y} = \sqrt{2} \int_{x_l}^{x_r} \frac{dx}{\sqrt{h - V(x)}},$$

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" **critical periods** = zeros$(T'(h)) \leq$ zeros$(m_k) =: n$, if $n < k$."

$\rightsquigarrow$ Chebyshev criterion for Abelian integrals [Grau, Mañosas, Villadelprat, '11]

Consider an analytic differential system satisfying

(H) The system has a center at the origin,
an analytic first integral of the form

$$H(x,y) = A(x) + B(x)y + C(x)y^2 \text{ with } A(0) = 0,$$

and its integrating factor K depends only on x.

Theorem (Garijo & Villadelprat, 2014)

*Under hypotheses **(H)** let $\mu_0 = -1$ and define for $i \geqslant 1$*

$$\mu_k := \left(\tfrac{1}{2} + \tfrac{1}{2k-3}\right)\mu_{k-1} + \frac{\sqrt{|C|}V}{(2k-3)K}\left(\frac{K\mu_{k-1}}{\sqrt{|C|}V'}\right)' \text{ and } \ell_k := \frac{K}{\sqrt{|C|}V'}\mu_k$$

If the number of zeros of $\mathscr{B}_\sigma(\ell_k)$ on $(0, x_r)$, counted with multiplicities, is $n \geqslant 0$ and it holds that $k > n$, then the number of critical periods of the center at the origin, counted with multiplicities, is at most n.

$\mathscr{B}_\sigma(\ell_k)$ denotes the σ-odd part of ℓ_k for an involution σ defined in terms of H.

The system

$$\begin{cases} w' = v, \\ v' = -\dfrac{A'(w) + \frac{1}{2}\, v^2}{w}, \end{cases}$$

has the first integral

$$H(w, v) := A(w) + \frac{w}{2} v^2.$$

To apply the criterion of [**GaVi14**], our system has to satisfy the following *hypotheses*:

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Period Annuli

$$\begin{cases} x' = y, \\ y' = -\dfrac{A'(x) + y^2}{2(x+\vartheta)}, \end{cases} \quad \text{where} \quad \begin{array}{l} A(x) = \frac{1}{2}x^2 - x^3, \\ \vartheta := \frac{1}{6}\left(\frac{2}{\sqrt{4+\frac{6\,\alpha}{\beta^2}}} - 1\right). \end{array} \tag{4}$$

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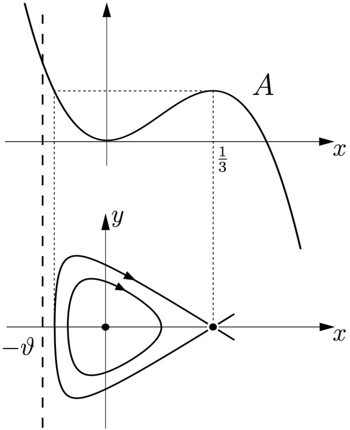

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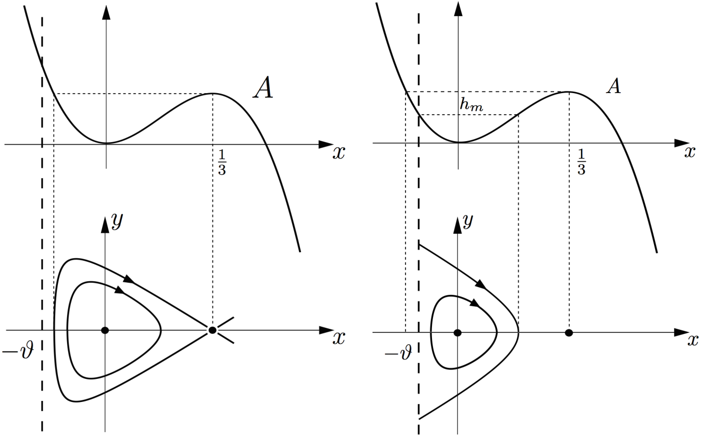
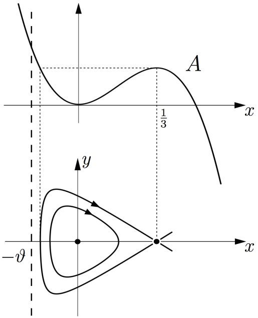


Proposition (1)

If $\vartheta \geqslant \frac{1}{6}$, then the period function of the center of system (4) *is monotonous increasing.*

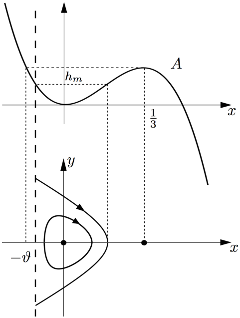
- Apply the criterion in [**GaVi14**] to deduce monotonicity.



Proposition (2)

For $\vartheta < \frac{1}{6}$ the period function of the center of (4) *is either monotonous decreasing for $\vartheta \in (0, \vartheta_1]$ or unimodal $\vartheta \in (\vartheta_1, 1/6)$, where $\vartheta_1 = -\frac{1}{10} + \frac{1}{15}\sqrt{6}$.*

- Apply criterion in **[GaVi14]** to obtain an upper bound for the critical periods.
- Compute the first period constants.
- Determine the sign of $T'(h)$ for $h \approx h_m$.



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Period function $T(h)$	monot.	≤ 1 crit. per.	≤ 2 crit. per.

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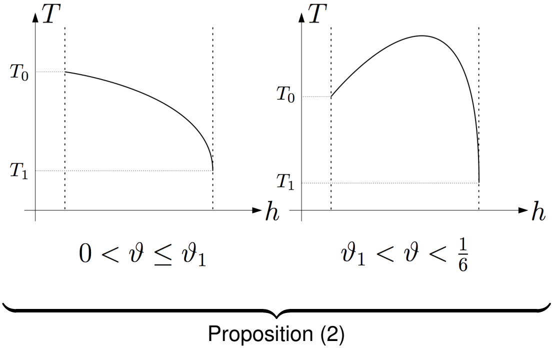
$\Downarrow$ [**GaVi14**], THM A			
Period function $T(h)$	monot.	≤ 1 crit. per.	≤ 2 crit. per.
$\Downarrow$ $h \approx 0, h \approx h_m$			
Period function $T(h)$	monot. decreasing		1 crit. period

- $T'(h) \lesseqgtr 0$ for $\vartheta \lesseqgtr \vartheta_1$ near $h = 0$.
- $T'(h) \longrightarrow -\infty$ as $h \to h_m$.

Sketch of the graph of the period function $T(h)$ of the center of (4):



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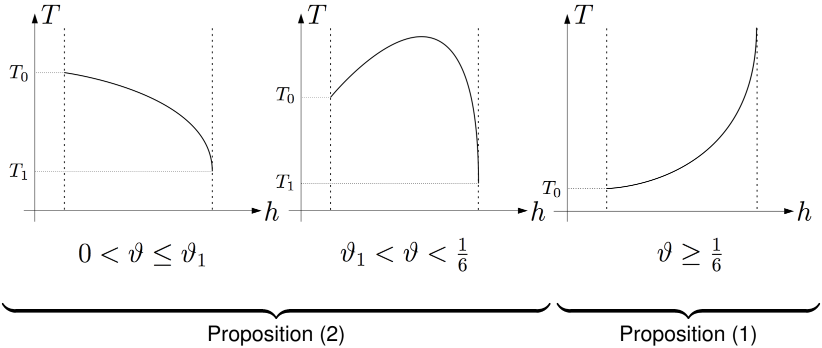

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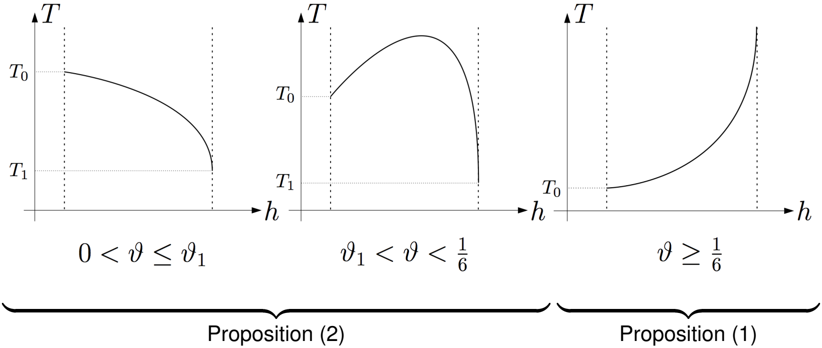
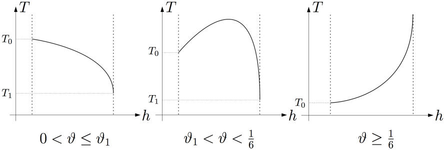


$$T_0 = 2\pi\sqrt{2\vartheta}, \quad T_1 = 2\ln\left(\frac{\sqrt{(2\vartheta+1)(1-6\vartheta)}}{1+6\vartheta-4\sqrt{\vartheta(1+3\vartheta)}}\right) > 0$$

Summary

wave lenght $\boldsymbol{\lambda}$ of φ	$=$	period $\boldsymbol{T}$ of γ_φ
wave height $\boldsymbol{a}$ of φ	$=$	$\boldsymbol{\ell(h_\varphi)}$
$\{\varphi_a\}_{a\in(0,a_M)}$	$\leftrightarrow$	$\{\gamma_h\}_{h\in(h_0,h_1)}$



Theorem

Given $c \neq -k$, there exist real numbers $r_1 < r_{b_1} < r_{b_2} < r_2$ such that the ***Camassa-Holm equation*** *has smooth periodic TWS satisfying*

$$\varphi''(\varphi - c) + \frac{(\varphi')^2}{2} + r + (c - 2k)\,\varphi - \frac{3}{2}\varphi^2 = 0,$$

if and only if $r \in (r_1, r_2)$. The set of ***smooth periodic TWS*** *form a continous family $\{\varphi_a\}_a$ parametrized by the wave height a.*

The wave length $\lambda = \lambda(a)$ of φ_a satisfies the following:

- *If $r \in (r_1, r_{b_1}]$, then $\lambda(a)$ is monotonous increasing.*
- *If $r \in (r_{b_1}, r_{b_2})$, then $\lambda(a)$ has a unique critical point (maximum).*
- *If $r \in [r_{b_2}, r_2)$, then $\lambda(a)$ is monotonous decreasing.*