Solitary waves of a class of Green-Naghdi type systems

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Based on a joint work with Erik Wahlén, Lund University and Vincent Duchêne, University of Rennes

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Background

Green-Naghdi (GN) equations

$$\begin{cases} \partial_t \zeta + \partial_x w & = 0 \\ \partial_t \left(\frac{w}{h} + Q[\zeta]w \right) + \partial_x \zeta + \frac{1}{2} \partial_x \left(\frac{w^2}{h^2} \right) & = \partial_x (R[\zeta, w]) \end{cases}$$

$h = 1 + \zeta$

$$\begin{aligned} Q[\zeta]w &= -\frac{1}{3h} \partial_x \left\{ h^3 \partial_x \left\{ \frac{w}{h} \right\} \right\} \\ R[\zeta, w] &= \frac{w}{3h^2} \partial_x \left\{ h^3 \partial_x \left\{ \frac{w}{h} \right\} \right\} + \frac{1}{2} \left(h \partial_x \left\{ \frac{w}{h} \right\} \right)^2 \end{aligned}$$

Model equation for shallow water.

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- (Miyata 85,87) generalized the (GN) equations to a two layer setting.
- (Lannes, Ming 2015) showed that the (GN)-equations overestimates the Kelvin-Helmholtz instabilities, meaning that the threshold for the velocity jump across the interface is smaller for (GN) then for the full Euler equations.
- (Duchêne, Israwi, Talhouk, 2015) suggested a modified (GN) system of equations which solves the problem with the Kelvin-Helmholtz instabilities, and which has the same dispersion relation as the full Euler equations.

Current work

$$\begin{cases} \partial_t \zeta + \partial_x w & = 0 \\ \partial_t \left(\frac{w}{h} + Q^F[\zeta]w\right) + \partial_x \zeta + \frac{1}{2}\partial_x \left(\frac{|w|^2}{h^2}\right) & = \partial_x (R^F[\zeta, w]) \end{cases}$$

$h = 1 + \zeta$

$$Q^F[\zeta]w = -\frac{1}{3h}\partial_x F\left\{h^3 \partial_x F\left\{\frac{w}{h}\right\}\right\}$$

$$R^F[\zeta, w] = \frac{w}{3h^2}\partial_x F\left\{h^3 \partial_x F\left\{\frac{w}{h}\right\}\right\} + \frac{1}{2}\left(h\partial_x F\left\{\frac{w}{h}\right\}\right)^2$$

and

$$\widehat{F\{\phi\}}(\xi) = F(\xi)\hat{\phi}(\xi).$$

Current work

Admissible class of Fourier multipliers

1. $F(\xi) = F(|\xi|)$ and $0 \leq F(\xi) \leq 1$.
2. $F(0) = 1, \; F'(0) = 0$.
3. There exists $0 \leq \theta < 1$ and $c, c' > 0$ such that

$$c|\xi|^{-\theta} \leq F(\xi) \leq c'|\xi|^{-\theta}, \text{ for } |\xi| >> 1.$$

Examples:

- $F = 1$ yields the original (GN) equation.
- $$F(\xi) = \sqrt{\frac{3}{\xi \tanh(\xi)} - \frac{3}{\xi^2}},$$
  yields a system with the same dispersion relation as the full Euler equations. Suggested by (Duchêne, Israwi, Talhouk, 2015).

Traveling waves

Traveling wave equation:

$$\zeta = c^2 \left(\frac{\zeta}{h} - \frac{1}{3h^2} \partial_x F \left\{ h^3 \partial_x F \left\{ \frac{\zeta}{h} \right\} \right\} - \frac{\zeta^2}{2h^2} + \frac{1}{2} \left(h \partial_x F \left\{ \frac{\zeta}{h} \right\} \right)^2 \right),$$

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$h = 1 + \zeta$.

Constrained minimization problem

$$\mathcal{E}(\zeta) = \int_{\mathbb{R}} \frac{\zeta^2}{1+\zeta} + \frac{(1+\zeta)^3}{3}\left(\partial_x F\left\{\frac{\zeta}{1+\zeta}\right\}\right)^2.$$

Consider

$$\mathrm{argmin}_{\zeta\in\Omega\subset X}\{\mathcal{E}(\zeta),\ \|\zeta\|_{L^2}^2 = q\},$$

$X \subset H^1(\mathbb{R})$, Ω open subset of X and $q \in (0, q_0)$. The solutions of this minimization problem will satisfy

$$\mathrm{d}\mathcal{E}(\zeta) + 2\alpha\zeta = 0,$$

where α is a Lagrange multiplier. This is the traveling wave equation with $\alpha = -\frac{1}{c^2}$.

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- Show that there exist solutions to the corresponding periodic problem.
- Construct a special minimizing sequence for the problem on the real line by using the minimizers from the periodic problem.
- Conclude by using the concentration compactness principle.

Penalized periodic problem

Penalization function: $\varphi : [0, R) \to [0, \infty)$ such that

$$\begin{aligned} \varphi(t) &= 0,\ 0 \le t \le \frac{R}{2}, \\ \lim_{t \nearrow R} \varphi(t) &= \infty, \\ \varphi'(t) &\le M_1 \varphi(t)^{a_1} + M_2 \varphi(t)^{a_2},\ 0 < a_1 < 1,\ a_2 > 0,\ M_1, M_2 > 0. \end{aligned}$$

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$$\lim_{t \nearrow R} \varphi(t) = \infty,$$
$$\varphi'(t) \le M_1 \varphi(t)^{a_1} + M_2 \varphi(t)^{a_2},\ 0 < a_1 < 1,\ a_2 > 0,\ M_1, M_2 > 0.$$

$$\mathcal{E}_{P,\varphi}(\zeta) := \varphi(\|\zeta\|_{H_P^1}^2) + \underbrace{\int_{-\frac{P}{2}}^{\frac{P}{2}} \frac{\zeta^2}{1+\zeta} + \frac{(1+\zeta)^3}{3} \left(\partial_x F \left\{ \frac{\zeta}{1+\zeta} \right\} \right)^2}_{:= \mathcal{E}_P(\zeta)}$$

where

$$\zeta \in V_{P,q,R} := \{\zeta \in H_P^1,\ \|\zeta\|_{L_P^2}^2 = q,\ \|\zeta\|_{H_P^1}^2 < R\}$$

$$H_P^s = \{u \in L_P^2,\ \|u\|_{H_P^s}^s := \sum \left(1 + \frac{4\pi^2 k^2}{P^2}\right)^s |\hat{u}_k|^2 < \infty\}.$$

Penalized periodic problem

Want to solve the minimization problem

$$\text{argmin}_{\zeta \in V_{P,q,R}} \mathcal{E}_{P,\varphi}(\zeta)$$

Lemma

The functional $\mathcal{E}_{P,\varphi}$ is weakly lower semi continuous, bounded from below and $\mathcal{E}_{P,\varphi} \to \infty$ as $\|\zeta\|_{H_1^P} \nearrow R$. In particular it has a minimizer $\zeta_P \in V_{P,q,R}$ which satisfies

$$2\varphi'(\|\zeta_P\|^2_{H^1_P})(\zeta_P - \partial_x^2 \zeta_P) + d\mathcal{E}_P(\zeta_P) + 2\alpha_P \zeta_P = 0,$$

for some Lagrange multiplier $\alpha_P(\zeta_P) \in \mathbb{R}$.

Periodic problem

Want to show that $\zeta_P \in V_{P,q,\frac{R}{2}}$.

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Lemma

The inequality

$$\|\zeta_P\|^2_{H^1_P} \leq cq$$

holds uniformly over the minimizers of $\mathcal{E}_{P,\varphi}$ over $V_{P,q,R}$, where $q \in (0, q_0),\ P \geq P_q$.

Choose q_0 small enough so that $\zeta_P \in V_{P,q,\frac{R}{2}}$.

Minimizers of the periodic problem

Theorem

There exists $R > 0, q_0 > 0$ *such that for any* $q \in (0, q_0)$ *one can define* $P_q > 0$ *so that the following holds. For each* $P \geq P_q$ *there exist* $\zeta_P \in V_{P,q,\frac{R}{2}}$ *such that*

$$\mathcal{E}_P(\zeta_P) = \inf_{\zeta \in H_P^1} \left\{ \mathcal{E}_P(\zeta),\ \|\zeta\|_{L_P^2}^2 = q,\ \|\zeta\|_{H_P^1}^2 < \frac{R}{2} \right\}.$$

Special minimizing sequence

A special minimizing sequence for the problem on the real line can be constructed from the minimizers of the periodic problem.

Theorem

There exists $q_0 > 0$ *such that for any* $q \in (0, q_0)$ *one can define* $\alpha < 0$ *and a sequence* $\{\zeta_n\}$ *satisfying*

$$\|\zeta_n\|_{L^2}^2 = q, \;\; \|\zeta_n\|_{H^1}^2 \leq cq, \;\; \lim_{n\to\infty} \|d\mathcal{E}(\zeta_n) + 2\alpha\zeta_n\|_{H^1} = 0,$$

and

$$\lim_{n\to\infty} \mathcal{E}(\zeta_n) = I_q := \inf_{\zeta\in H^1} \{\mathcal{E}(\zeta), \;\; \|\zeta\|_{L^2}^2 = q\}.$$

Want to extract a convergent subsequence from his minimizing sequence.

Concentration Compactness principle

Theorem

Any sequence $\{e_n\}_{n\in\mathbb{N}} \in L^1(\mathbb{R})$ *of non-negative functions such that* $\int_{\mathbb{R}} e_n = q > 0$ *admits a subsequence, denoted again* $\{e_n\}_{n\in\mathbb{N}}$*, for which one of the following holds.*

- *(Vanishing) For each* $r > 0$*, one has*

$$\lim_{n\to\infty}\left(\sup_{x\in\mathbb{R}}\int_{Br(x)} e_n\ dx\right) = 0.$$

- *(Dichotomy) There are real sequences* $\{x_n\}_{n\in\mathbb{N}},\ \{M_n\}_{n\in\mathbb{N}},\ \{N_n\}_{n\in\mathbb{N}}$ *and* $\lambda \in (0, q)$ *such that*

$$M_n\ , N_n \to \infty,\ \frac{M_n}{N_n} \to 0,\ \int_{B_{M_n}(x_n)} e_n\ dx \to \lambda \text{ and } \int_{B_{N_n}(x_n)} e_n\ dx \to \lambda.$$

- *(Concentration) There exists a sequence* $\{x_n\}_{n\in\mathbb{N}}$ *such that for each* $\epsilon > 0$ *there exist* $r > 0$ *such that*

$$\int_{B_r(x_n)} e_n\ dx \geq q - \epsilon$$

Existence of minimizer

- Apply concentration compactness to $\{\zeta_n^2\}_n^\infty$, where $\zeta_n{}_n^\infty$ is the special minimizing sequence, and assume that concentration holds.
- Then there exists a sequence $\{x_n\}$ such that

$$\|\eta_n\|_{L^2(|x|>r)}^2 < \epsilon, \text{ where } \eta_n = \zeta_n(\cdot + x_n).$$

- Have that $\|\eta_n\|_{H^1}^2 \leq cq$, so we may assume that $\eta_n \rightharpoonup \eta$ in $H^1(\mathbb{R})$, which implies that $\eta_n \to \eta$ in $L^2(|x| \leq r)$.
- Follows that $\eta_n \to \eta$ in $L^2(\mathbb{R})$.
- By interpolating we then have that $\eta_n \to \eta$ in $H^s(\mathbb{R})$ for all $s \in [0, 1)$.
- In particular this is true for $s = 1 - \theta$ and $\mathcal{E}(\zeta) \sim \|\zeta\|_{H^{1-\theta}}^2$, and so $I_q = \lim_{n\to\infty} \mathcal{E}(\eta_n) = \mathcal{E}(\eta)$.

Excluding dichotomy

- Show that the map $q \to I_q$, where $I_q = \inf_{\zeta \in H^1} \{\mathcal{E}(\zeta) \ : \ \|\zeta\|_{L^2}^2 = q\}$ is strictly subadditive.
- Assume that dichotomy occurs. Can the construct sequences $\eta_n^{(1)}$, $\eta_n^{(2)}$ such that

$$\left\|\eta_n^{(1)}\right\|_{L^2}^2 = \lambda, \ \left\|\eta_n^{(2)}\right\|_{L^2}^2 = q - \lambda,$$
$$\lim_{n\to\infty} (\mathcal{E}(\eta_n) - \mathcal{E}(\eta_n^{(1)}) - \mathcal{E}(\eta_n^{(2)})) = 0$$

  where $\eta_n(x) = \zeta_n(x + x_n)$.
- By definition:

$$\mathcal{E}(\tilde{\eta}_n^{(1)}) \geq I_\lambda \text{ and } \mathcal{E}(\tilde{\eta}_n^{(2)}) \geq I_{q-\lambda}.$$

Excluding dichotomy

Use these properties to obtain a contradiction:

$$\begin{aligned} I_q &< I_\lambda + I_{q-\lambda} \\ &\le \lim_{n\to\infty} \mathcal{E}(\tilde{\eta}_n^{(1)}) + \mathcal{E}(\tilde{\eta}_n^{(2)}) \\ &= \lim_{n\to\infty} \mathcal{E}(\eta_n) \\ &= I_q \end{aligned}$$

Thanks!