Higher Order Dispersion From Phase Dynamics

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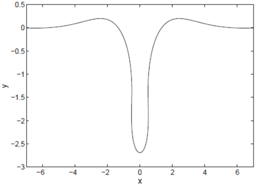
Theoretical and Computational Aspects of Nonlinear Surface Waves
BIRS 2016

Motivation



(a) Solitary wave from 5th order KdV model (Părău and Guyenne, 2015)

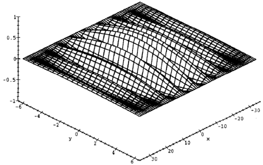


Fig. 6. – Solitary wave of elevation for $q_0 = 0.75$, $\varepsilon = -0.01$.

(b) Travelling solitary wave from 5th order KP model (Hărăguş-Courcelle and Il'ichev, 1998)

Aim: Find out how fifth order dispersion arises from modulational arguments, apply to water wave problems and other physically interesting systems.

Outline

Abstract Setup

- Multisymplectic formulation and relative equilibria
- Linear operator and solvability
- Conservation laws in multisymplectic settings
- Jordan chain theory

Modulation and Summary of Asymptotics

Examples

- Shallow water plate problem
- Higher Order NLS equation

Future Work/ Next Steps

Euler-Lagrange Equations and Conservation Laws I

Start from the *Multisymplectic* form of the Euler-Lagrange equations,

$$\mathbf{M}Z_t + \mathbf{J}Z_x + \mathbf{K}Z_y = \nabla S(Z), \qquad \mathbf{M}^T = -\mathbf{M},\ \mathbf{J}^T = -\mathbf{J},\ \mathbf{K}^T = -\mathbf{K}.$$

Assume the existence of a single phase *relative equilibrium* (e.g. periodic travelling wave) solution,

$$Z(x,y,t) = \widehat{Z}(\theta; k, m, \omega), \quad \theta = kx + my + \omega t + \theta_0 \tag{1.1}$$

for wavenumbers k, m and frequency ω.

Euler-Lagrange Equations and Conservation Laws II

Define the linear operator about $\widehat{Z}$ as

$$\mathbf{L} = \mathrm{D}^2 S(\widehat{Z}) - (\omega \mathbf{M} + k\mathbf{J} + m\mathbf{K})\partial_\theta,$$

which leads to the results

$$\mathbf{L}\widehat{Z}_\theta = 0, \quad \mathbf{L}\widehat{Z}_k = \mathbf{J}\widehat{Z}_\theta$$

Assuming the kernel is no larger, solvability of systems in this setting requires that

$$\mathbf{L}F = G \quad \text{is solvable when} \quad \langle\!\langle \widehat{Z}_\theta, G \rangle\!\rangle = 0,$$

for suitable inner product $\langle\!\langle \cdot, \cdot \rangle\!\rangle$.

Euler-Lagrange Equations and Conservation Laws III

Define the quantities

$$A(Z) = \frac{1}{2}\langle\langle Z, \mathbf{M} Z_\theta \rangle\rangle, \quad B = \frac{1}{2}\langle\langle Z, \mathbf{J} Z_\theta \rangle\rangle, \quad C = \frac{1}{2}\langle\langle Z, \mathbf{K} Z_\theta \rangle\rangle$$

which form the conservation law

$$A_t + B_x + C_y = 0.$$

Evaluate these along $\widehat{Z}$ to give these as function of k m and ω:

$$A(\widehat{Z}),\ B(\widehat{Z}),\ C(\widehat{Z}) \ \equiv\ \mathscr{A}(k,m,\omega),\ \mathscr{B}(k,m,\omega),\ \mathscr{C}(k,m,\omega).$$

The derivatives of these relate to solvability requirements and coefficients in the final equation.

Jordan Chain Theory I

The theory admits Jordan chains of the form

$$\mathbf{L}\xi_1 = \mathbf{0}, \quad \mathbf{L}\xi_{i+1} = \mathbf{J}\xi_i.$$

As has been seen, $\xi_1 = \widehat{Z}_\theta$, $\xi_2 = \widehat{Z}_k$ and a third element exists when

$$\mathscr{K}_2 = \langle\!\langle \mathbf{J}\xi_1, \xi_2 \rangle\!\rangle = -\langle\!\langle \widehat{Z}_\theta, \mathbf{J}\widehat{Z}_k \rangle\!\rangle = -\mathscr{B}_k = 0.$$

The chain is always even in length (since **L**'s zero eigenvalue is even) and so fifth element exists when

$$\mathscr{K}_4 = \langle\!\langle \mathbf{J}\widehat{Z}_\theta, \xi_4 \rangle\!\rangle = 0.$$

Consequence: No third order dispersion.

Jordan Chain Theory II

There is also a mixed chain of the form

$$\mathbf{L}\zeta_1 = \mathbf{J}\widehat{Z}_m + \mathbf{K}\widehat{Z}_k, \quad \mathbf{L}\zeta_{i+1} = \mathbf{J}\zeta_i + \mathbf{K}\xi_{i+2},$$

which will lead to mixed dispersion. The first element exists when

$$\mathscr{M}_0 = -\langle\!\langle \widehat{Z}_\theta, \mathbf{J}\widehat{Z}_m + \mathbf{K}\widehat{Z}_k \rangle\!\rangle = \mathscr{B}_m + \mathscr{C}_k = 0.$$

This chain is also of even length, and in the analysis the relevant coefficient that emerges is

$$\mathscr{M}_2 = -\langle\!\langle \widehat{Z}_\theta, \mathbf{J}\zeta_2 + \mathbf{K}\xi_4 \rangle\!\rangle,$$

which may or may not vanish, depending on the application considered.

Modulation Approach

Idea: Find relative equilibrium $\widehat{Z}$, then consider an ansatz by perturbing the independent variables (modulation) as

$$Z = \widehat{Z}(\theta + \varepsilon^3 \phi(X, Y, T), k + \varepsilon^4 q(X, Y, T), m + \varepsilon^6 r(X, Y, T), \omega + \varepsilon^8 \Omega(X, Y, T)) + \varepsilon^5 \sum_{n=0}^{\infty} \varepsilon^n W_n(\theta, X, Y, T) \quad (2.2)$$

with $X = \varepsilon x$, $Y = \varepsilon^3 y$, $T = \varepsilon^5 t$ and $\varepsilon \ll 1$. Method is to substitute the ansatz into the Euler-Lagrange equation, expand around $\varepsilon = 0$ and solve at each order.

Strengths of the approach:

- Do asymptotics on general Euler-Lagrange equations <u>once</u>, then result applies to all systems that can be put in that form (providing relevant criterion met).
- Coefficients are related to properties of the basic state - can be determined *a-priori* and are simple to compute.

Summary of Key Step in Asymptotics

- Everything is trivial until $\mathcal{O}(\varepsilon^5)$ (by ansatz construction), at which stage we must solve

$$\mathbf{L}W_0 = q_X \mathbf{J}\widehat{Z}_k,$$

  which can be done when $\mathscr{B}_k = 0$.

- The mixed chain emerges at $\mathcal{O}(\varepsilon^6)$:

$$\mathbf{L}(W_1 - q_{XX}\xi_4) = q_Y(\mathbf{J}\widehat{Z}_m + \mathbf{K}\widehat{Z}_k),$$

  which is solvable when $\mathscr{B}_m + \mathscr{C}_k = 0$.

- At $\mathcal{O}(\varepsilon^7)$ the third order dispersive term in X emerges, which vanish when $\langle\langle \widehat{Z}_\theta, \mathbf{J}\xi_4 \rangle\rangle = -\mathscr{K}_4 = 0$. If it doesn't then regular KP is most suitable model.[1]

- Solvability at $\mathcal{O}(\varepsilon^9)$ leads to the fifth order KP

$$\Big((\mathscr{A}_k + \mathscr{B}_\omega) q_T + \mathscr{B}_{kk} q q_X + \mathscr{M}_2 q_{XXY} + \mathscr{K}_6 q_{XXXXX} \Big)_X + \mathscr{C}_m q_{YY} = 0$$

[1] T. J. Bridges. "Emergence of dispersion in shallow water hydrodynamics via modulation of uniform flow". In: *J. Fluid Mech.* 761 (2014), R1.

Summary of Result

The key result is that the fifth order KP equation is a suitable model when

$$\boxed{\mathscr{K}_2 = \mathscr{K}_4 = 0, \quad \mathscr{B}_m + \mathscr{C}_k = 0,}$$

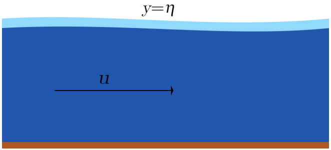
and therefore three conditions need to be met. If the system has the symmetry $y \mapsto -y$, then the last is automatic (by choosing $m = 0$), but a consequence is that $\mathscr{M}_2 = 0$ also.

Example 1 - Biharmonic Elastic Sheet

Consider the potential shallow water system with linear (biharmonic) elastic plate on the surface

$$\boxed{\begin{gathered} \eta_t + \nabla \cdot (\eta \nabla \phi) = 0, \\ \phi_t + \frac{1}{2} |\nabla \phi|^2 + g\eta + \frac{D}{\rho} \nabla^4 \eta = R, \end{gathered}} \tag{3.3}$$

for velocity potential ϕ, free surface height η, rigidity constant D and Bernoulli constant R.



Basic solution is constant velocity, thus

$$\phi = \theta, \quad \eta = \eta_0 = g^{-1}\left(R - \omega - \frac{k^2 + m^2}{2}\right).$$

Criticality and Emergence of KP-5

The conservation law components are

$$\mathscr{A} = \eta_0, \quad \mathscr{B} = k\eta_0, \quad \mathscr{C} = m\eta_0$$

Criticality in $\mathscr{B}$ occurs when

$$\eta_0 - \frac{k^2}{g} = 0 \quad \text{(Froude number criticality).}$$

Fifth order dispersion then happens automatically. Transverse symmetry imposes $m = 0$.
The modulation theory gives the resulting fifth order KP as

$$\boxed{\left(q_T + \frac{3}{2}qq_X - \frac{Dk}{2\rho g}q_{XXXXX}\right)_X + \frac{k}{2}q_{YY} = 0, \quad k = \pm\sqrt{g\eta_0}.}$$

Example II: Higher Order NLS

Consider the model

$$i\psi_t + \nabla^2\psi + \frac{1}{2}\lambda\nabla^4\psi + \psi - |\psi|^2\psi = 0,$$

which has been proposed to model higher order dispersive effects in Maxwell's equations[2].
Relative equilibrium associated with the $SO(2)$ symmetry group gives the solution $\psi = \psi_0 e^{i\theta}$ with

$$|\psi_0|^2 = 1 - (k^2 + m^2) + \frac{1}{2}\lambda(k^2 + m^2)^2.$$

[2] V.I. Karpman. "Influence of high-order dispersion on self-focusing. I. Qualitative investigation". In: *Phys. Lett. A* 160.6 (1991), pp. 531–537.

Criticality and Emergence

The conservation laws along the relative equilibrium are given by

$$\mathscr{A} = \frac{1}{2}|\psi_0|^2, \quad \mathscr{B} = k\big(1 - \lambda(k^2 + m^2)\big)|\psi_0|^2, \quad \mathscr{C} = m\big(1 - \lambda(k^2 + m^2)\big)|\psi_0|^2.$$

The first and third order dispersion terms vanish when $k \approx -0.533$, $m = 0$, $\lambda \approx -1.796$ (solving numerically). In which case we the resulting fifth order KP is

$$\big(q_T + aqq_X + bq_{XXXXX}\big)_X + cq_{YY} = 0$$

with

$$a \approx 4.569, \quad b \approx -1.483, \quad c \approx 0.139$$

Next Steps

- If $\mathscr{B}_{kk} = 0$, one expects terms like

$$\ldots + \left[\frac{1}{2}\mathscr{B}_{kkk} q^2 q_X + \partial_k \mathscr{K}_4 (q q_{XXX} + 2 q_X q_{XX})\right]_X + \ldots,$$

  which appear in Craig and Groves[3] and Parau and Guyenne[4].
- Find examples where $\mathscr{M}_2 \neq 0$!
- Undertake a similar analysis for multiple conservation laws (which have more parameters and so fifth order models are more attainable) - can potentially lead to coupled 5th order models.

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[3] W. Craig and M. D. Groves. "Hamiltonian long-wave approximations to the water-wave problem". In: *Wave motion* 19.4 (1994), pp. 367–389.

[4] P. Guyenne, E. I. Părău, et al. "Asymptotic Modeling and Numerical Simulation of Solitary Waves in a Floating Ice Sheet". In: *The Twenty-fifth International Offshore and Polar Engineering Conference*. International Society of Offshore and Polar Engineers. 2015.

Thanks for listening!