Lefschetz properties for Artinian Gorenstein algebras presented by quadrics

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In the paper *Gorenstein algebras presented by quadrics*, Migliore and Nagel proposed the following two conjectures.

Conjecture

(Injective Conjecture) *For any Artinian Gorenstein algebra presented by quadrics, defined over a field $\mathbb{K}$ of characteristic zero, and of socle degree at least three, there exists $L \in A_1$, such that, the multiplication map $\bullet L : A_1 \to A_2$ is injective.*

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(WLP Conjecture) *Any Artinian Gorenstein algebra presented by quadrics, over a field $\mathbb{K}$ of characteristic zero, has the Weak Lefschetz Property.*

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Let $\mathbb{K}$ be an infinite field, $R = \mathbb{K}[x_1, \ldots, x_n]$ be the polynomial ring in n indeterminates and $Q = \mathbb{K}[X_1, \ldots, X_n]$ the ring of differential operators. Let $I \subset Q$ be a homogeneous ideal. It is well known that $A = Q/I$ is a Gorenstein algebra if and only if there exists a homogeneous $f \in R$ such that $I = \mathrm{Ann}_Q(f)$. In order to get I generated by quadrics we start choosing f to be square free, so $(X_1^2, \ldots, X_n^2) \subset I$.

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We deal with a very special kind of f that comes from the theory of Gordan-Noether-Perazzo-Permutti of forms with vanishing hessian. The forms we consider are naturally bigraded and the separation of variables of $R = \mathbb{K}[x_1, \ldots, x_n, u_1, \ldots, u_m]$ has an important role, as we will see in the sequel. Let $A = \bigoplus_{i=0}^{d} A_i$, $A_d \neq 0$ be a standard bigraded Artinian Gorenstein algebra with $A_k = \bigoplus_{i=0}^{k} A_{(i,k-1)}.A_{(d_1,d_2)} \neq 0$ for some d_1, d_2 such that $d_1 + d_2 = d$, we call (d_1, d_2) the socle bidegree of A. Since $A_k^* \simeq A_{d-k}$ and since duality is compatible with direct sum, we get $A^*_{(i,j)} \simeq A_{(d_1-i, d_2-j)}$.

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Let $f \in R_{(d_1,d_2)}$ be a bihomogeneous polynomial of total degree $d = d_1 + d_2$, then $I = \text{Ann}_Q(f) \subset Q$ is a bihomogeneous ideal and $A = Q/I$ is a standard bigraded Artinian Gorenstein algebra of socle bidegree (d_1, d_2) and codimension $r = m + n$.

Let $f \in R_{(d_1,d_2)}$ be a bihomogeneous polynomial of bidegree (d_1, d_2) and let A be the associated bigraded algebra of socle bidegree (d_1, d_2). For all $i > d_1$ or $j > d_2$ we get $I_{(i,j)} = Q_{(i,j)}$. As consequence, we have the following decomposition for all A_k:

$$A_k = \bigoplus_{i+j=k, i \leq d_1, j \leq d_2} A_{(i,j)}.$$

Furthermore, for $i < d_1$ and $j < d_1$, the evaluation map $Q_{i,j} \to A_{(d_1-i,d_2-j)}$ given by $\alpha \mapsto \alpha(f)$ provides the following short exact sequence:

$$0 \to I_{(i,j)} \to Q_{(i,j)} \to A_{(d_1-i,d_2-j)} \to 0.$$

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Definition

Let $A = Q/\operatorname{Ann}_Q(f) = \bigoplus_{k=0}^{d} A_k$ be a standard graded Artinian Gorenstein $\mathbb{K}$ algebra of socle degree d. Let $i \leq j \leq \frac{d}{2}$ be two integers and let $B_k = \{\alpha_1, \ldots, \alpha_s\}$ and $B_l = \{\beta_1, \ldots, \beta_t\}$ be bases of the $\mathbb{K}$-vector spaces A_k and A_l respectively. The (mixed) hessian matrix of f of order (k, l) is the matrix:

$$\operatorname{Hess}_f^{(k,l)} = (\alpha_i(\beta_j(f)))_{s \times t}.$$

We denote $\operatorname{Hess}_f^k := \operatorname{Hess}_f^{(k,k)}$, $\operatorname{hess}_f^k := \det(\operatorname{Hess}_f^k)$ and $\operatorname{hess}_f := \operatorname{hess}_f^1$.

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The following result is a generalization of a Theorem due to Maeno - Watanabe in the paper *Lefschetz elements of artinian Gorenstein algebras and Hessians of homogeneous polynomials*.

Theorem

(Hessian Lefschetz criterion)

Let $A = Q/\operatorname{Ann}_Q(f)$ *be a standard graded Artinian Gorenstein algebra of codimension* r *and socle degree* d *and let* $L = a_1x_1 + \ldots + a_rx_r \in A_1$*, such that* $f(a_1, \ldots, a_r) \neq 0$*. The map* $\bullet L^{l-k} : A_k \to A_l$*, for* $k \leq \frac{d}{2}$*, has maximal rank if and only if the (mixed) Hessian matrix* $\operatorname{Hess}_f^{(k,d-l)}(a_1, \ldots, a_r)$ *has maximal rank.*

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Example

Consider the cubic hypersurface $X = V(f) \subset \mathbb{P}^7$, given by

$$f = \begin{vmatrix} x_0 & x_1 & x_2 \\ x_3 & x_4 & x_5 \\ x_6 & x_7 & 0 \end{vmatrix} \in \mathbb{K}[x_0, \ldots, x_7].$$

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After a linear change of coordinates we can rewrite

$$f = x_1u_1u_2 + x_2u_2u_3 + x_3u_3u_4 + x_4u_4u_1 \in R = \mathbb{K}[\underline{x}, \underline{u}].$$

The associated algebra $A = Q/I$ with $I = \mathrm{Ann}(f)$ does not have the WLP, in fact the map $\bullet L : A_1 \to A_2$ is not injective for any $L \in A_1$. A is presented by quadrics. Indeed:

$$I = \begin{array}{c}(u_4^2, u_2u_4, x_2u_4, x_1u_4, u_3^2, u_1u_3, x_4u_3, x_1u_3, u_2^2, x_4u_2, x_3u_2,\\ x_2u_2 - x_3u_4, x_1u_2 - x_4u_4, u_1^2, x_4u_1 - x_3u_3, x_3u_1, x_2u_1,\\ x_1u_1 - x_2u_3, x_4^2, x_3x_4, x_2x_4, x_1x_4, x_3^2, x_2x_3, x_1x_3, x_2^2, x_1x_2, x_1^2).\end{array}$$

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The previous example motivates the study of bihomogeneous Perazzo polynomials of bidegree $(1, d-1)$. They can be written in the form

$$f = x_1 g_1 + \ldots + x_n g_n,$$

where $g_i \in \mathbb{K}[u_1, \ldots, u_m]_{d-1}$.We say that f is of *monomial square free type* if all g_i are square free monomials.The associated algebra, $A = Q/\operatorname{Ann}_Q(f)$, is bigraded, has socle bidegree $(1, d-1)$ and we assume that $I_1 = 0$, so $\operatorname{codim} A = m + n$.

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where $g_i \in \mathbb{K}[u_1, \ldots, u_m]_{d-1}$.We say that f is of *monomial square free type* if all g_i are square free monomials.The associated algebra, $A = Q/\operatorname{Ann}_Q(f)$, is bigraded, has socle bidegree $(1, d-1)$ and we assume that $I_1 = 0$, so $\operatorname{codim} A = m + n$.

Let us consider the homogeneous (or pure) simplicial complex $\mathcal{K} = \mathcal{K}(u)$ of dimension $d-1$ whose facets are given by the monomials g_i. The 0-skeleton will be referred as vertex set and we write $V = \{u_1, \ldots, u_m\}$. We identify the 1-skeleton with a simple graph $\mathcal{K}_1 = (V, E)$, hence the 1-faces are called edges. Since $X_i(f) = g_i$, we identify each facet g_i with the differential operator X_i. We denote by e_k the number of k-faces, hence $e_0 = m$ and $e_{d-1} = n$.

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Theorem

Let f be a bihomogeneous polynomial of monomial square-free type, $\mathcal{K}$ the simplicial complex and A the algebra of codimension $m+n$ and socle bidegree $(1, d-1)$. Then

$$A = \bigoplus_{k=0}^{d} A_k$$

where $A_k = A_{(0,k)} \oplus A_{(1,k-1)}$. Moreover, $A_{(0,k)}$ has a basis identified with the k faces of $\mathcal{K}$, hence $\dim A_{(0,k)} = e_k$. By duality, $A^*_{(1,k-1)} \simeq A_{(0,d-k)}$, and a basis for $A_{(1,k-1)}$ can be chosen by taking, for each $d-k$-face of $\mathcal{K}$, a monomial $X_i\tilde{G}_i$ such that $X_i\tilde{G}_i(f)$ represents this $d-k$-face.In particular, the Hilbert vector of A is given by $h_k = \dim A_k = e_k + e_{d-k}$. Furthermore, $I = \mathrm{Ann}_Q(f)$ is a binomial ideal generated by $(X_1, \ldots, X_n)^2$, by all the monomials in U_i that do not represent faces of $\mathcal{K}$, by all monomials X_iF_i where f_i does not represent a subface of g_i and by all binomials $X_i\tilde{G}_i - X_j\tilde{G}_j$ where $g_i = \tilde{g}_i g_{ij}$ and $g_j = \tilde{g}_j g_{ij}$ and g_{ij} represents a common subface of g_i, g_j.

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$$A = \bigoplus_{k=0}^{d} A_k \text{ where } A_k = A_{(0,k)} \oplus A_{(1,k-1)}.$$

Moreover, $A_{(0,k)}$ has a basis identified with the k faces of $\mathcal{K}$, hence $\dim A_{(0,k)} = e_k$. By duality, $A^*_{(1,k-1)} \simeq A_{(0,d-k)}$, and a basis for $A_{(1,k-1)}$ can be chosen by taking, for each $d-k$-face of $\mathcal{K}$, a monomial $X_i\tilde{G}_i$ such that $X_i\tilde{G}_i(f)$ represents this $d-k$-face. In particular, the Hilbert vector of A is given by $h_k = \dim A_k = e_k + e_{d-k}$.

Furthermore, $I = \mathrm{Ann}_Q(f)$ is a binomial ideal generated by $(X_1, \ldots, X_n)^2$, by all the monomials in U_i that do not represent faces of $\mathcal{K}$, by all monomials X_iF_i where f_i does not represent a subface of g_i and by all binomials $X_i\tilde{G}_i - X_j\tilde{G}_j$ where $g_i = \tilde{g}_i g_{ij}$ and $g_j = \tilde{g}_j g_{ij}$ and g_{ij} represents a common subface of g_i, g_j.

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Definition

Let $\mathcal{K}$ be a homogeneous simplicial complex of dimension $d-1$. We say that $\mathcal{K}$ is facet connected if for any pair of facets F, F' of $\mathcal{K}$ there exists a sequence of facets, $F_0 = F, F_1, \ldots, F_s = F'$ such that $F_i \cap F_{i+1}$ is a $(d-2)$-face. We say that $\mathcal{K}$ is upper closed if for all complete subgraphs $H = K_l \subset \mathcal{K}_1$ there is a l-face $F \in \mathcal{K}_l$ such that H is the first skeleton of F. In particular $\mathcal{K}_1$ does not contain any K_d.

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Theorem

Let $f \in \mathbb{K}[x_1, \ldots, x_n, u_1, \ldots, u_m]_{(1,d-1)}$ *be a bihomogeneous polynomial of monomial square-free type, let* $\mathcal{K}$ *be the associated simplicial complex and let* $A = Q/\operatorname{Ann}_Q(f)$ *be the standard bigraded Artinian Gorenstein algebra.* *A is presented by quadrics if and only if* $\mathcal{K}$ *is facet connected and upper closed.*

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Definition

Let $2 \leq a_1 \leq \ldots \leq a_{d-1}$ be integers, the Turan complex of order $a_1, \ldots, a_{d-1}$, $\mathcal{K} = \mathcal{TK}(a_1, \ldots, a_{d-1})$, is the Homogeneous simplicial complex whose facets set is the cartesian product $\pi = \prod_{i=1}^{d-1} \{1, 2, \ldots, a_i\}$. The Turan polynomial of order $a_1, \ldots, a_{d-1}$ it the multihomogeneous polynomial

$$f = f_{\mathcal{K}} = \sum_{\alpha \in \pi} x_\alpha u_\alpha \in R = \mathbb{K}[x_\alpha, u_{(i,j_i)}]_{\alpha \in \pi, 1 \leq i \leq d-1, 1 \leq j_i \leq a_i},$$

where $\alpha = (j_1, \ldots, j_{d-1}) \in \pi$ and $u_\alpha = u_{(1,j_1)} \cdots u_{(d-1,j_{d-1})}$. The Turan algebra of order $(a_1, \ldots, a_{d-1})$ is $TA(a_1, \ldots, a_{d-1}) = A_{\mathcal{K}} = Q/\operatorname{Ann}_Q(f)$.

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The number of k-faces of a Turan complex is $e_k = s_k$ where $s_k = s_k(a_1, \ldots, a_{d-1})$ is the symmetric function of order k. By Theorem 8, the Hilbert vector of the Turan algebra $TA(a_1, \ldots, a_{d-1})$ is given by $h_k = s_k + s_{d-k}$. It is easy to verify that Turan complexes are facet connected and upper closed.

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Theorem

Let $A = TA(a_1, \ldots, a_{d-1})$ be the Turan algebra of order $(a_1, \ldots, a_{d-1})$ with $2 \leq a_1 \leq a_2 \leq \ldots \leq a_{d-1}$. Then A is presented by quadrics and for all $L \in A_1$ the map $\bullet L : A_1 \to A_2$ is not injective. Furthermore, if $a_1 \approx \ldots \approx a_{d-1}$ are large enough, then $\mathrm{Hilb}(A)$ is not unimodal in the first step, that is, $\dim A_1 > \dim A_2$.

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