Togliatti systems and artinian ideals failing the Weak Lefschetz Property

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Lefschetz Properties and Artinian Algebras
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Outline

Joint work with R.M. Miró-Roig and G. Ottaviani

Osculating spaces

K algebraically closed field, $\mathrm{char}(K) = 0$

$X \subset \mathbb{P}^N$ projective variety of dimension n
$x \in X$ a smooth point

- X has an affine local parametrization $\Phi(t_1, \ldots, t_n)$ in formal power series in a neighbourhood of x
- $t_1, \ldots, t_n$ local parameters
- $x = \Phi(0, \ldots, 0)$

The vector tangent space $T_x X$, in differential geometric sense, is generated by the partial derivatives vectors $\Phi_{t_1}(0), \ldots, \Phi_{t_n}(0)$. The s-th osculating space $T_x^{(s)}X$, $s \geq 1$, is generated by all partial derivatives of order $\leq s$.

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Rational varieties

If X is a rational variety, there is a rational parametrization

$$\phi : \mathbb{P}^n \cdots \xrightarrow{[F_0,\dots,F_N]} X \subset \mathbb{P}^N$$

$F_i \in S := K[x_0, \dots, x_n]_d$

The embedded s-th osculating space $\mathbb{T}_x^{(s)} X$ is generated by all partials of order s of ϕ (Euler).

Definition

1. The expected dimension of $\mathbb{T}_x^{(s)} X$ is $\min\{N, \binom{n+s}{s} - 1\}$
2. X satisfies δ Laplace equations of order s if at a general smooth point x

$$\dim \mathbb{T}_x^{(s)} X = \binom{n+s}{s} - 1 - \delta$$

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Examples

1. Ruled varieties: the parametrization can be chosen so that one of the variables appears at most at degree 1 in the components of ϕ.

2. Curves: at a general point the osculating space always has the expected dimension.

3. Togliatti surface (Eugenio Togliatti, 1929, 1946) $X \subset \mathbb{P}^5$, rational surface parametrized by cubics, special projection of the del Pezzo sextic surface $S \subset \mathbb{P}^6$.

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Togliatti surface

The parametrization is:

$$\phi : \mathbb{P}^2 \dashrightarrow \mathbb{P}^5$$
$$\phi : [x, y, z] \to [x^2y, x^2z, xy^2, xz^2, y^2z, yz^2]$$

The Laplace equation:
$$x^2\phi_{xx} - xy\phi_{xy} - xz\phi_{xz} + y^2\phi_{yy} - yz\phi_{yz} + z^2\phi_{zz} = 0$$

Geometric interpretation: consider

$v_3 : (\mathbb{P}^2)^* \to \mathbb{P}^9 = \mathbb{P}(K[a, b, c]_3)$ the triple Veronese embedding
$v_3 : l[a, b, c] \to l^3$

then project $v_3(\mathbb{P}^2)$ first from the plane $\langle a^3, b^3, c^3\rangle$ and get the del Pezzo surface S, then from the point abc, that belongs to all its osculating spaces.

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1946: Togliatti classifies all the projections of $v_3(\mathbb{P}^2)$ satisfying at least one Laplace equation of order 2

2007: Brenner - Kaid:
if char $K = 0$, an ideal $I = (x^3, y^3, z^3, f(x, y, z))$, $\deg f = 3$, fails WLP if and only if $f \in (x^3, y^3, z^3, xyz)$.

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Apolarity, or Macaulay-Matlis duality

$I \subset S = K[x_0, \ldots, x_n]$ homogenous ideal

$\mathcal{D} = K[y_0, \ldots, y_n]$ S-module with the product given by differentiation:
$FD := F(\frac{\partial}{\partial y_0}, \ldots, \frac{\partial}{\partial y_n})(D)$

$I^{-1} = \{D \in \mathcal{D} \mid FD = 0 \;\; \forall F \in I\}$, graded S-submodule of $\mathcal{D}$: Macaulay inverse system of I

Conversely: given $M \subset \mathcal{D}$ graded S-submodule, $\mathrm{Ann}(M) \subset S$ is a homogenous ideal.

Bijection:

$$\{\text{homogeneous artinian ideals of } S\}$$
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$$\{\text{graded finitely generated } S-\text{submodules of } \mathcal{D}\}$$

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Apolar varieties

If I is monomial, generated by monomials all of degree d, I^{-1} can be seen inside S, generated by the monomials of degree d not in I.

If I is artinian: $I = (F_1, \dots, F_r)$, $F_1, \dots, F_r$ of degree d, then $(I^{-1})_d$ is a linear system of hypersurfaces of degree d.
We have maps:

$\phi_{(I^{-1})_d} : \mathbb{P}^n \dashrightarrow X \subset \mathbb{P}^N$, $N = \binom{n+d}{d} - 1 - r$

X rational variety projection of $v_d(\mathbb{P}^n)$ from $\langle F_1, \dots, F_r \rangle$,

$\phi_{I_d} : \mathbb{P}^n \to Y \subset \mathbb{P}^{r-1}$ is a morphism.

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Togliatti systems

Theorem (MMO, 2013)

$I \subset S$ homogeneous artinian ideal generated by $F_1, \dots, F_r$ of degree d. Let $r \leq \binom{n+d-1}{n-1}$. The following are equivalent:

1. *I fails WLP in degree $d-1$;*
2. *$F_1, \dots, F_r$ become linearly dependent in $S/(L)$, for any linear form L;*
3. *X satisfies at least one Laplace equation of order $d-1$.*

The ideals I as in the Theorem are called Togliatti systems.

Remark

The assumption on r means that the Laplace equations satisfied by X are not trivial.
For example, if $n = 2$ $r \leq d + 1$.

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Minimal and smooth Togliatti systems

Definition

A Togliatti system I is called:

- monomial if I can be generated by monomials;
- minimal if I does not contain any smaller Togliatti system;
- smooth if X is a smooth variety.

Goal: classify the minimal smooth Togliatti systems.

Reformulation of Togliatti's result: if $n = 2$, $d = 3$, the only smooth Togliatti system is $I = (x^3, y^3, z^3, xyz)$.

Remark

If $I \subset S = K[x_0, \dots, x_n]$ is artinian monomial generated in degree d, then I contains $x_0^d, \dots, x_n^d$.

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Togliatti systems of cubics, $n = 3$

Theorem (MMO, 2013)

If $n = 3$, $d = 3$, the only monomial minimal smooth Togliatti systems are

1. $(x^3, y^3, z^3, t^3, xyz, xyt, xzt, yzt)$ *the triple embedding of* $\mathbb{P}^3$ *blown up at 4 general points, then suitably projected from a* $\mathbb{P}^3$ *- truncated simplex*
2. $(x^3, y^3, z^3, t^3, x^2y, xy^2, xzt, yzt)$ *the triple embedding of* $\mathbb{P}^3$ *blown up at 2 points and a line, then suitably projected from a* $\mathbb{P}^1$
3. $(x^3, y^3, z^3, t^3, x^2y, xy^2, z^2t, zt^2)$ *the triple embedding of* $\mathbb{P}^3$ *blown up at 2 skew lines.*

This answers to a conjecture of G. Ilardi (2006).

Monomial Togliatti systems of cubics

In [MMO] there is also a class of examples and a conjecture for monomial minimal smooth Togliatti systems of cubics with $n \geq 3$.

Michałek and Miró-Roig classify monomial minimal smooth Togliatti systems of quadrics and cubics, proving the conjecture (2016).

Monomial Togliatti systems, any d

For $d > 3$ the situation is much more intricate.

[MM, 2016]
Let $\mu(I) =$ minimal number of generators of I.

- Computation of the minimal and maximal bound on $\mu(I)$ for $I \in \mathcal{T}(n, d)$ and $I \in \mathcal{T}^s(n, d)$
- Classification on the border and near the border
- Existence results in the admissible range

where:
$\mathcal{T}(n, d)$: minimal monomial Togliatti systems $K[x_0, \dots, x_n]$ generated in degree d

$\mathcal{T}^s(n, d)$: minimal smooth monomial Togliatti systems in $K[x_0, \dots, x_n]$ generated in degree d

Toric varieties

$I \subset S = K[x_0, \dots, x_n]$ homogeneous artinian generated by monomials of degree d

I^{-1} inverse system also contained in S

The variety $X = \phi_{(I^{-1})_d}(\mathbb{P}^n)$ is a toric projective variety and can be studied with combinatorial methods [Gelfand - Kapranov - Zelevinski].

Definition

Δ_n standard simplex in $\mathbb{R}^{n+1}$ with coordinates $a_0, \dots, a_n$
Consider $d\Delta_n$ in the hyperplane $a_0 + \cdots + a_n = d$ identified with $\mathbb{R}^n$
Every point $(a_0, \dots, a_n)$ of $d\Delta_n \cap \mathbb{Z}^n$ with $a_0 + \cdots + a_n = d$ corresponds to a monomial $x_0^{a_0} \dots x_n^{a_n} \in S_d$.

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Polytopes

Given $I \subset S$ monomial artinian ideal generated in degree d:

- $A_I \subset d\Delta_n$ is the set of points corresponding to monomials of degree d not in I, i.e., in $(I^{-1})_d$.
- P_I is the convex hull of A_I: the polytope associated to I.

Example: Togliatti's surface

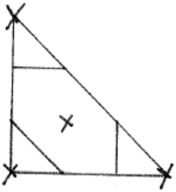

Figure : A_I is the punctured hexagon

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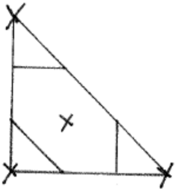

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Two Togliatti examples with $n = 3$, $d = 3$

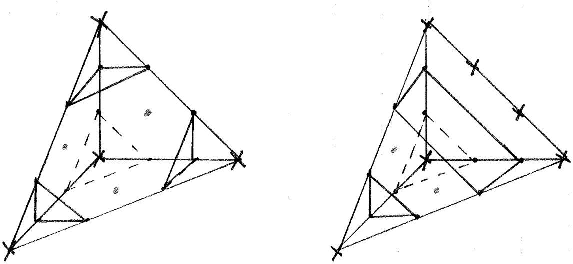

Figure : The truncated simplex and case 2

If $n > 3$ all monomial minimal smooth Togliatti systems correspond to a partition of the vertices (with some condition). We remove the corresponding faces and the centres of the remaining hexagons.

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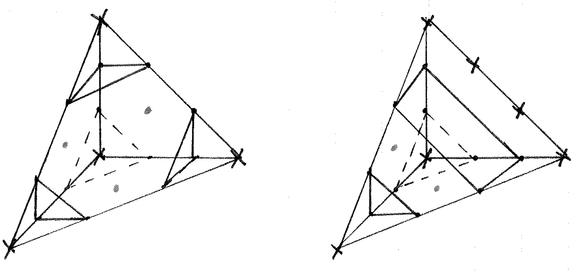

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Two facts

Theorem (Perkinson, 2000)

I is a Togliatti system if and only there exists a hypersurface of degree $d-1$ in $\mathbb{R}^n$ containing all points of A_I.

Moreover I is a minimal Togliatti system if and only if every such hypersurface does not contain any point of $d\Delta_n \setminus A_I$ except possibly some vertex.

Theorem (GKZ)

The toric variety X is smooth if and only if

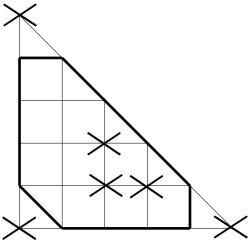
1. *translating every vertex v of P_I in the origin, and considering on any edge emanating from v the first point with integer coordinates, they form a $\mathbb{Z}$-basis of $\mathbb{Z}^n$;*
2. *technical condition. If $n = 2$ it says that every inner point in an edge of P_I must belong to A_I.*

Trivial Togliatti systems

Definition

A Togliatti system is trivial if I contains $x_0 F, \ldots, x_n F$ for some form F of degree $d-1$.

The hypersurface of degree $d-1$ containing all points of A_I is a union of hyperplanes. Here is a trivial smooth Togliatti system with $d = 5$.



A trivial monomial Togliatti system can have $2n+1$ or $2n+2$ generators.
For instance, with $d=4$:

- $2n+1$ generators: $(x_0^4, \dots, x_n^4, x_0^3x_1, \dots, x_0^3x_n)$, or
- $2n+2$ generators: $(x_0^4, \dots, x_n^4, x_0^2x_1x_2, x_0x_1^2x_2, x_0x_1x_2^2, x_0x_1x_2x_3, \dots, x_0x_1x_2x_n)$

Results

Define

$\mu(n,d) = \min\{\mu(I) \mid I \in \mathcal{T}(n,d)\}$

$\mu^s(n,d) = \min\{\mu(I) \mid I \in \mathcal{T}^s(n,d)\}$

Theorem (MM, 2016)

1. *For all* $n \geq 2$, $d \geq 4$

$$\mu(n,d) = \mu^s(n,d) = 2n+1.$$

2. *If* $\mu(I) = 2n+1$ *then either* I *is trivial, or* $n = 2$, $\mu(I) = 5$ *and, up to permutation of the coordinates, either*
   - $d = 4$, $I = (x^4, y^4, z^4, x^2yz, y^2z^2)$ *non-smooth, or*
   - $d = 5$, $I = (x^5, y^5, z^5, x^3yz, xy^2z^2)$ *smooth.*

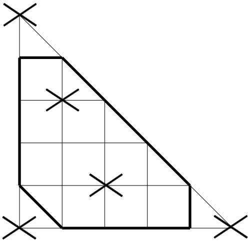


Figure : The smooth Togliatti system with $n = 2$, $d = 5$, $\mu(I) = 5$.

This rational surface X has inflection points, i.e. points where the dimension of $\mathbb{T}_x^{(s)}X$ decreases more than for general x, for $s \leq 4$.
The hypersurface of degree 4 containing A_I is irreducible.

Theorem (MM, 2016)

If $d \geq 4$ and I is a smooth monomial minimal Togliatti system with $\mu(I) = 2n + 2$, then either I is trivial, or $n = 2$, $\mu(I) = 6$. Moreover, up to permutation of the coordinates, either

- *$d = 5$, three cases, or*
- *$d = 7$, three cases.*

Picture for $n = 2$, $d \geq 4$

Define

$\rho(n,d) = \max\{\mu(I) \mid I \in \mathcal{T}(n,d)\}$

$\rho^s(n,d) = \max\{\mu(I) \mid I \in \mathcal{T}^s(n,d)\}$

Let $n = 2$. For any $d \geq 4$:

- $\mu(2,d) = \mu^s(2,d) = 5$;
- $\rho(2,d) = \rho^s(2,d) = d+1$;
- for any r with $5 \leq r \leq d+1$, there exists $I \in \mathcal{T}^s(2,d)$ with $\mu(I) = r$.

Picture for $n = 2$, $d \geq 4$

Define
$\rho(n, d) = \max\{\mu(I) \mid I \in \mathcal{T}(n, d)\}$
$\rho^s(n, d) = \max\{\mu(I) \mid I \in \mathcal{T}^s(n, d)\}$

Let $n = 2$. For any $d \geq 4$:

- $\mu(2, d) = \mu^s(2, d) = 5$;
- $\rho(2, d) = \rho^s(2, d) = d + 1$;
- for any r with $5 \leq r \leq d + 1$, there exists $I \in \mathcal{T}^s(2, d)$ with $\mu(I) = r$.

Gaps for $n \geq 3$

If $n \geq 3$, $d \geq 4$:

- monomial Togliatti systems with $\mu(I) = 2n+1, 2n+2$ are trivial;
- there is no smooth monomial minimal Togliatti system with $\mu(I) = 2n+3$;
- $\rho(n,d) = \binom{n+d-1}{n-1}$;
- if $n = 3$, for any $d \geq 4$ and r with $\mu(3,d) = 7 \leq r \leq \rho(3,d) = \binom{d+2}{2}$, there exists $I \in \mathcal{T}(3,d)$ with $\mu(I) = r$.

Thank you!

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