The Local Cut Lemma and Critical Hypergraphs

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New Trends in Graph Colouring
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What is the Local Cut Lemma?

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The Local Cut Lemma is a strengthening of the LLL that implies the combinatorial results obtained using the entropy compression method.
A hypergraph $\mathcal{H}$ is a pair $(V(\mathcal{H}), E(\mathcal{H}))$, where $V(\mathcal{H})$ is a finite set (whose elements are the vertices of $\mathcal{H}$) and $E(\mathcal{H})$ is a collection of nonempty subsets of $V(\mathcal{H})$ (called the edges of $\mathcal{H}$).
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A *hypergraph* \( H \) is a pair \( (V(H), E(H)) \), where \( V(H) \) is a finite set (whose elements are the *vertices* of \( H \)) and \( E(H) \) is a collection of nonempty subsets of \( V(H) \) (called the *edges* of \( H \)).

A hypergraph \( H \) is *true* if all its edges have size at least 3.

A *proper \( k \)-colouring* of \( H \) is a function \( f : V(H) \to \{1, \ldots, k\} \) such that \( |f(H)| \geq 2 \) for all \( H \in E(H) \) (i.e., there are no *monochromatic* edges).
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Question
What is the minimum number of edges in a \((k + 1)\)-critical true hypergraph on \(n\) vertices?

Theorem (Abbott–Hare 1989)
For every \(\varepsilon > 0\), there exists a \((k + 1)\)-critical true hypergraph with \(n\) vertices and at most \((k - 1 + \varepsilon)n\) edges.

Theorem (Kostochka–Stiebitz 2000)
Every \((k + 1)\)-critical true hypergraph with \(n\) vertices has at least \((k - 3k^2/3)n\) edges.
Critical hypergraphs

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I will show how to use the LCL to improve the Kostochka–Stiebitz result as follows:

**Theorem**

Every \((k + 1)\)-critical true hypergraph with \(n\) vertices has at least \((k - 4\sqrt{k})n\) edges.
I will show how to use the LCL to improve the Kostochka–Stiebitz result as follows:

**Theorem**

Every \((k + 1)\)-critical true hypergraph with \(n\) vertices has at least \((k - 4\sqrt{k})n\) edges.

The proof is almost the same as the proof of the Kostochka–Stiebitz theorem, with the LLL replaced by the LCL.
Say a vertex is “heavy” if it belongs to many edges (and the smaller the edge, the more weight it contributes).
Proof idea

Say a vertex is “heavy” if it belongs to many edges (and the smaller the edge, the more weight it contributes).

We iteratively remove “heavy” vertices from $\mathcal{H}$. 

Case 1: If at the end all the vertices have been removed, then we obtain a lower bound on $|E(\mathcal{H})|$. 

Case 2: If at the end we are left with a nonempty set $U$ of “light” vertices, then we can first colour $\mathcal{H} - U$ and then extend this colouring to $U$. 

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Let $v \in U \subseteq V(\mathcal{H})$. We say that a vertex $v$ is heavy in $U$ if

$$\sum_{H \ni v} w(|H \cap U|) \geq k - 4\sqrt{k},$$

where $w: \mathbb{N}_{\geq 1} \to \mathbb{R}_{> 0}$ is a weight function satisfying

$$\sum_{t=1}^{\infty} w(t) = 1.$$
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Algorithm

Set $U_0 := V(\mathcal{H})$. 
Construction

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Algorithm

Set $U_0 := V(\mathcal{H})$.

If $U_i$ contains a heavy vertex $v_i$, then set $U_{i+1} := U_i \setminus \{v_i\}$. 
First case: nothing is left

**Case 1:** This process ends with $U_n = \emptyset$. 
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Fix a proper $k$-colouring of $\mathcal{H} - U$ and extend it to a $k$-colouring of $\mathcal{H}$ by choosing a colour for each vertex in $U$ uniformly at random.
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Need to show that with positive probability, the resulting colouring is proper.
In (a simplified version of) the LCL, one is given a finite set $X$ and a random collection $\mathcal{A}$ of subsets of $X$ that is closed downwards. The goal is to show that $\Pr[X \in \mathcal{A}] > 0$. 

In our application, $X$ is $U$ and a subset $S \subseteq U$ belongs to $\mathcal{A}$ if and only if there is no monochromatic edge $H \subseteq S \cup U^c$. To apply the LCL, for every $S \subseteq X$ and $v \not\in S$, we have to specify a collection of “bad” random events $B(S, v)$ with the following property: If $S \{v\} \in \mathcal{A}$, but $S \not\in \mathcal{A}$, then at least one event in $B(S, v)$ has happened. In our case, set $B_H := \{H \text{ is monochromatic}\}$ and define $B(S, v) := \{B_H : v \in H \subseteq S \cup U^c\}$. 

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In our case, set $B_H := \{H \text{ is monochromatic}\}$ and define

$$\mathcal{B}(S, v) := \{B_H : v \in H \subseteq S \cup U^c\}.$$
Fix a parameter $\omega \in [1; +\infty)$. For each $B \in B(S, v)$, we require an upper bound on the following quantity:

$$\rho_\omega(B) := \min_{v \notin S' \subseteq S} \Pr[B | S' \in A] \cdot \omega^{|S \setminus S'|}.$$
Fix a parameter $\omega \in [1; +\infty)$. For each $B \in \mathcal{B}(S, \nu)$, we require an upper bound on the following quantity:

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**Case 1:** $H \not\subseteq U$. 

**Case 2:** $H \subseteq U$. 

Actually, we can do a little better. Fix some $u \in H \setminus \{\nu\}$. Then

$$\rho_\omega(B) \leq \Pr[B \mid (S \setminus H) \cup \{u\} \in \mathcal{A}] \cdot \omega^{|H| - 1}.$$
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Applying the Local Cut Lemma

**Theorem (The Local Cut Lemma)**

If there is \( \omega \in [1; +\infty) \) such that for every \( v \in S \subseteq X \) we have

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\omega \geq 1 + \sum_{B \in B(S, v)} \rho_{\omega}(B),
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In our case, we need

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\omega \geq 1 + \sum_{v \in H \not\subseteq U} \frac{\omega|H \cap U|}{k|H \cap U|} + \sum_{v \in H \subseteq U} \frac{\omega|H|^{-1}}{k|H|^{-1}}.
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The rest is just a straightforward computation.
Thank you!
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$$\geq (k - 4\sqrt{k})n,$$

as desired.