The Local Cut Lemma and Critical Hypergraphs

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New Trends in Graph Colouring
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What is the Local Cut Lemma?

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The Local Cut Lemma is a strengthening of the LLL that implies the combinatorial results obtained using the entropy compression method.

A *hypergraph* $\mathcal{H}$ is a pair $(V(\mathcal{H}), E(\mathcal{H}))$, where $V(\mathcal{H})$ is a finite set (whose elements are the *vertices* of $\mathcal{H}$) and $E(\mathcal{H})$ is a collection of nonempty subsets of $V(\mathcal{H})$ (called the *edges* of $\mathcal{H}$).

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A *proper k-colouring* of $\mathcal{H}$ is a function $f \colon V(\mathcal{H}) \to \{1, \ldots, k\}$ such that $|f(H)| \geq 2$ for all $H \in E(\mathcal{H})$ (i.e., there are no *monochromatic* edges).

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A hypergraph $\mathcal{H}$ is $(k+1)$*-critical* if it is not k-colourable, but all its proper subhypergraphs are.

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For every $\varepsilon > 0$, there exists a $(k+1)$-critical true hypergraph with n vertices and at most $(k-1+\varepsilon)n$ edges.

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Theorem (Kostochka–Stiebitz 2000)

Every $(k+1)$-critical true hypergraph with n vertices has at least $(k-3k^{2/3})n$ edges.

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The proof is almost the same as the proof of the Kostochka–Stiebitz theorem, with the LLL replaced by the LCL.

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Construction

Let $\mathcal{H}$ be a $(k+1)$-critical true hypergraph on n. We want to show that $|E(\mathcal{H})| \geq (k - 4\sqrt{k})n$.

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Let $v \in U \subseteq V(\mathcal{H})$. We say that a vertex v is *heavy* in U if

$$\sum_{H \ni v} w(|H \cap U|) \geq k - 4\sqrt{k},$$

where $w \colon \mathbb{N}_{\geq 1} \to \mathbb{R}_{>0}$ is a weight function satisfying

$$\sum_{t=1}^{\infty} w(t) = 1.$$

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Algorithm

Set $U_0 := V(\mathcal{H})$.

If U_i contains a heavy vertex v_i, then set $U_{i+1} := U_i \setminus \{v_i\}$.

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Fix a proper k-colouring of $\mathcal{H} - U$ and extend it to a k-colouring of $\mathcal{H}$ by choosing a colour for each vertex in U uniformly at random.

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Need to show that with positive probability, the resulting colouring is proper.

Setting up the Local Cut Lemma

In (a simplified version of) the LCL, one is given a finite set X and a random collection $\mathcal{A}$ of subsets of X that is closed downwards. The goal is to show that $\Pr[X \in \mathcal{A}] > 0$.

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In our application, X is U and a subset $S \subseteq U$ belongs to $\mathcal{A}$ if and only if there is no monochromatic edge $H \subseteq S \cup U^c$.

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In our case, set $B_H := \{H \text{ is monochromatic}\}$ and define

$$\mathcal{B}(S, v) := \{B_H : v \in H \subseteq S \cup U^c\}.$$

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Fix a parameter $\omega \in [1; +\infty)$. For each $B \in \mathcal{B}(S, v)$, we require an upper bound on the following quantity:

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Applying the Local Cut Lemma

Theorem (The Local Cut Lemma)

If there is $\omega \in [1; +\infty)$ such that for every $v \in S \subseteq X$ we have

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then $\Pr[X \in \mathcal{A}] > 0$.

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$$\omega \geq 1 + \sum_{v \in H \not\subseteq U} \frac{\omega^{|H \cap U|}}{k^{|H \cap U|}} + \sum_{v \in H \subseteq U} \frac{\omega^{|H|-1}}{k^{|H|-1}}.$$

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The rest is just a straightforward computation.

Thank you!

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$$\geq (k - 4\sqrt{k})n,$$

as desired.