DP-Coloring

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Recall that Vizing introduced list coloring trying to prove an approximation to the Behzad-Vizing Conjecture that the total chromatic number of any graph with maximum degree Δ is at most $\Delta + 2$.

The plan was: given a set D of $\Delta + 3$ colors, color from D the vertices of G, and then every edge will have a list of $\Delta + 1$ available colors.

The plan did not work as planned, but the new notion (introduced also by Erdős, Rubin and Taylor) turned out to be valuable and interesting. Some properties of it are very close to those of the ordinary coloring, and some are quite different.

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One well-known application of list coloring is the Fleischner-Stiebitz proof of the cycle-plus-triangles problem by Erdős.

A Gallai forest is a graph in which every block is either a complete graph or an odd cycle.

Theorem 1 [Gallai, 1963] If $k \geq 3$ and G is a k-critical graph, then the subgraph of G induced by the vertices of degree $k-1$ is a Gallai forest.

A list L for a graph G is a degree list if $|L(v)| \geq \deg_G(v)$ for all $v \in V(G)$.

Theorem 2 [Borodin, 1976; Erdős–Rubin–Taylor, 1979] Let G be a connected graph and let L be a degree list assignment for G. If G is not L-colorable, then G is a Gallai tree; furthermore, $|L(u)| = \deg_G(u)$ for all $u \in V(G)$ and if $u, v \in V(G)$ are two adjacent non-cut vertices, then $L(u) = L(v)$.

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Theorem 3 [Plesnevič and Vizing, 1965] A graph G has a k-coloring if and only if the Cartesian product $G \square K_k$ contains an independent set of size $|V(G)|$, i.e., $\alpha(G \square K_k) = |V(G)|$.

Pre-definition

Given a list L for G, the vertex set of the auxiliary graph $H = H(G, L)$ is $\{(v, c) : v \in V(G) \text{ and } c \in L(v)\}$, and two distinct vertices (v, c) and (v', c') are adjacent in H if and only if either $c = c'$ and $vv' \in E(G)$, or $v = v'$.

Since $V(H)$ is covered by $|V(G)|$ cliques, $\alpha(H) \le |V(G)|$. If H has an independent set I with $|I| = |V(G)|$, then, for each $v \in V(G)$, there is a unique $c \in L(v)$ such that $(v, c) \in I$. And the same color c is not chosen for any two adjacent vertices. So the map $f : V(G) \to \mathbb{Z}_{>0}$ defined by $(v, f(v)) \in I$ is an L-coloring of G.

Also, if G has an L-coloring f, then the set $\{(v, f(v)) : v \in V(G)\}$ is an independent set of size $|V(G)|$ in H.

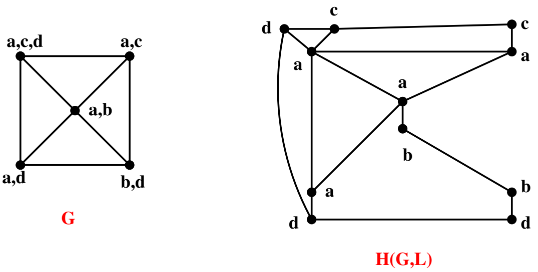

Figure: A graph G with a list L and a cover for (G, L).

Definition

Let G be a graph. A *cover* of G is a pair (L, H), where L is an assignment of pairwise disjoint sets to the vertices of G and H is a graph with vertex set $\bigcup_{v \in V(G)} L(v)$, satisfying the following:

1. For each $v \in V(G)$, $H[L(v)]$ is a complete graph.
2. For each $uv \in E(G)$, the edges between $L(u)$ and $L(v)$ form a matching (possibly empty).
3. For each distinct $u, v \in V(G)$ with $uv \notin E(G)$, no edges of H connect $L(u)$ and $L(v)$.

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Let G be a graph and (L, H) be a cover of G. An *(L, H)-coloring* of G is an independent set $I \subseteq V(H)$ of size $|V(G)|$. G is *(L, H)-colorable* if it admits an (L, H)-coloring.

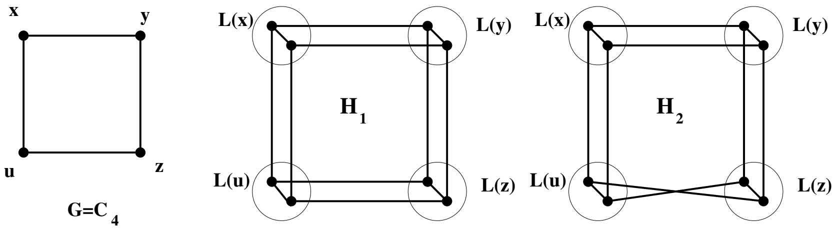

Figure: Graph C_4 and two covers of it such that C_4 is (L, H_1)-colorable but not (L, H_2)-colorable.

The *DP-chromatic number*, $\chi_{DP}(G)$, is the minimum k such that G is (L, H)-colorable for each choice of (L, H) with $|L(v)| \geq k$ for all $v \in V(G)$.

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5. $\chi_{DP}(G) > \frac{d/2}{\ln(d/2)}$ for every G with average degree d. (A. Bernshteyn)

6. $\chi_{DP}(G) \le C\frac{d}{\ln d}$ for every triangle-free G with maximum degree d. (A. B.)

Multigraphs

Let G be a multigraph. A *cover* of G is a pair (L, H), where L is an assignment of pairwise disjoint sets to the vertices of G and H is a graph with vertex set $\bigcup_{v \in V(G)} L(v)$ such that

1. For each $v \in V(G)$, $H[L(v)]$ is a complete graph.
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Figure: $\chi_{DP}(K_2^3) = 4$.

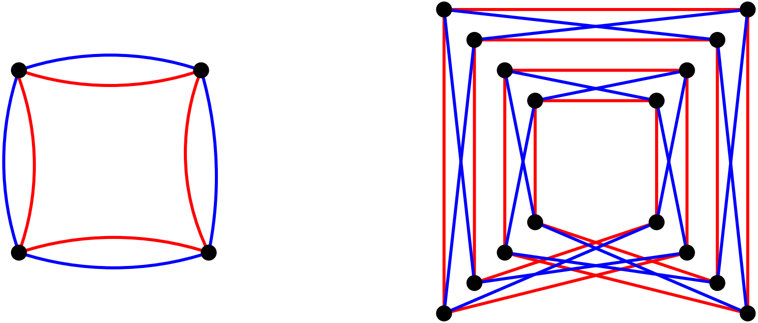

Figure: $\chi_{DP}(C_4^k) = 2k + 1$.

Theorem 4 [Bernshteyn-Pron-A.K., 2016] Let G be a connected multigraph. Then G is not DP-degree-colorable if and only if each block of G is one of the graphs K_n^k, C_n^k for some n and k.

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Corollary 5 [B-P-K] Let $k \geq 4$ and let G be a DP-k-critical graph distinct from K_k. Then

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Theorem 6 [Dirac, 1957] Let $k \geq 4$ and let G be a k-critical graph distinct from K_k. Set $n := |V(G)|$ and $m := |E(G)|$. Then

$$2m \geq kn + k - 3.$$

For $k \geq 4$, a graph G is *k-Dirac* if $V(G)$ can be partitioned into three subsets V_1, V_2, V_3 so that

(a) $|V_1| = k - 1$, $|V_2| = k - 2$, $|V_3| = 2$;

(b) the graphs $G[V_1]$ and $G[V_2]$ are complete;

(c) each $y_i \in V_1$ is adjacent to exactly one $z_j \in V_3$, and each $z_j \in V_3$ has a neighbor in V_1;

(d) each $x_i \in V_2$ is adjacent to both $z_j \in V_3$; and

(e) G has no other edges.

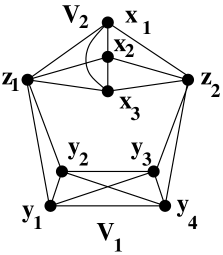

Figure: A 5-Dirac graph.

Let $\mathcal{D}_k$ denote the family of all k-Dirac graphs.

Theorem 7 [Dirac, 1974] Let $k \geq 4$ and let G be a k-critical graph distinct from K_k. Set $n := |V(G)|$ and $m := |E(G)|$. Then

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Theorem 8 [A.K.-Stiebitz, 2002] Let $k \geq 4$ and let L be a list assignment for G such that G is L-critical and $|L(u)| = k - 1$ for all $u \in V(G)$. Suppose that G does not contain a clique of size k. Set $n := |V(G)|$ and $m := |E(G)|$. Then

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Question [A.K.-Stiebitz, 2002] Does Theorem 7 hold for list coloring?

Theorem 9 [B-K] Let $k \geq 4$, G be a graph and let (L, H) be a cover of G such that G is (L, H)-critical and $|L(u)| = k - 1$ for all $u \in V(G)$. Suppose that G does not contain a clique of size k. Set $n := |V(G)|$ and $m := |E(G)|$. If $G \notin \mathcal{D}_k$, then

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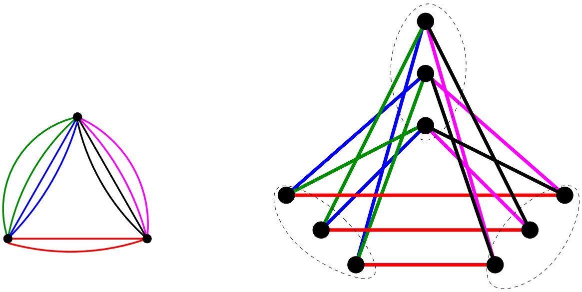

Figure: A DP-7-critical multigraph.

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Does there exist $0 < \alpha < 1$ such that for every n and every $k > \alpha n$, each n-vertex DP-k-critical graph G has a spanning complete bipartite subgraph.