Tight lower bounds for the complexity of multicoloring

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χ: Minimum number of colors to ensure that:

$$\chi : \begin{cases} 
|C| = b \\
|D| = b \\
C \cap D = \emptyset 
\end{cases}$$

$$\lim_{b \to \infty} \chi_b = \chi_f$$
\[ c \neq d \]

\[ \chi: \text{Minimum number of colors to ensure that:} \]

\[ |C| = b \quad |D| = b \quad C \cap D = \emptyset \]

\[ \lim_{b \to \infty} \chi_b = \chi_f \]
$\chi$: Minimum number of colors to ensure that:

\[ C_x \subseteq D_y \implies c \neq d \]

where $x$ and $y$ are vertices, and $c$ and $d$ are colors.
\( \chi \): Minimum number of colors to ensure that:

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\[ \frac{c}{d} \quad \Rightarrow \quad c \neq d \]

\( x \quad y \)
\( \chi \): Minimum number of colors to ensure that:

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\begin{aligned}
C_x \stackrel{c}{\longrightarrow} D_y & \Rightarrow c \neq d \\
C_x \cap D_y & = \emptyset
\end{aligned}
\]

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\begin{aligned}
|C| & = b \\
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\end{aligned}
\]
\( \chi \): Minimum number of colors to ensure that:

\[
\begin{align*}
\chi \colon & \Rightarrow c \neq d \\
C \cup D \Rightarrow & \begin{cases} 
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C \cap D = \emptyset
\end{cases}
\end{align*}
\]
$\chi$: Minimum number of colors to ensure that:

\[
\begin{align*}
\circ C \quad \& \quad \circ D \quad \Rightarrow \quad c \neq d \\
\times x \quad \& \quad \times y
\end{align*}
\]

$\chi_b$: Minimum number of colors to ensure that:

\[
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C \cap D = \emptyset
\end{cases}
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\]

\[
\lim_{b \to \infty} \frac{\chi_b}{b} = \chi_f
\]
”Is $G$ $k$-colorable?”
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- $k \leq 1$: Easy ✓
- $k = 2$: Easy ✓
- $k \geq 3$: NP-hard

"Is $G$ $a:b$-colorable?"

- $a < 2$: Easy ✓
- $a = 2$: Easy ✓
- $a \geq 2b + 1$: NP-hard (Hell, Nešetřil '90)
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Complexity

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NP-hard? :(

Exponential Time Hypothesis (Impagliazzo, Paturi '99)
There is $\epsilon > 0$ such that $3$-SAT cannot be solved in $O^*(2^{\epsilon \cdot n})$ time.

Theorem (Dell, Husfeldt, Wahlén '10)
For any $k \geq 3$, there is $\alpha > 0$ such that $k$-Coloring cannot be solved in $O^*(2^{\alpha \cdot n})$ time unless ETH fails.

Theorem (Björklund, Husfeldt '06)
$k$-Coloring can be solved in $O^*(2^n)$ time.
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Our result

Theorem (Nederlof '08)

\textit{\textbf{a:b-Coloring}} \textit{can be solved in} $\mathcal{O}^*((b + 1)^n)$ \textit{time}.
Our result

**Theorem (Nederlof '08)**

\[a:b\text{-Coloring} \text{ can be solved in } \mathcal{O}^*((b + 1)^n) \text{ time.}\]

**Theorem (B., Kowalik, Pilipczuk, Socała, Wrochna '16)**

There is \(\alpha > 0\) such that, for appropriate ranges of values, \(a:b\text{-Coloring} \text{ cannot be solved in } \mathcal{O}^*((b + 1)^{\alpha \cdot n}) \text{ time unless ETH fails.}\)
The reduction

Fix $a$, $b$. 
The reduction

Fix $a$, $b$. Main idea: **compress** an instance $\phi$ of 3-SAT on $n$ variables and $m$ clauses into the $a:b$-coloring of a graph $G$ on $O\left(\frac{m+n}{\log b}\right)$ vertices.
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**Sparsification Lemma (Tovey '84)**

*We can assume that in $\phi$, every variable belongs to at most 4 clauses.*
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**Sparsification Lemma (Tovey ’84)**

*We can assume that in $\phi$, every variable belongs to at most 4 clauses.*

We can also relax $a:b$-coloring: every vertex is assigned

- an integer $\in \{1, \ldots, b\}$ (**number of colors to receive**) and
- a subset of $\{1, \ldots, a\}$ (**colors it’s allowed to take**).
The reduction (2)

\( v_1, \ldots, v_n: \) variables of \( \phi \).
\( c_1, \ldots, c_m: \) clauses of \( \phi \).
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$v_1, \ldots, v_n$: variables of $\phi$.

$c_1, \ldots, c_m$: clauses of $\phi$.

- **Groups** of variables of size $\log b$: $V_1, \ldots, V_{\frac{n}{\log b}}$
- **Groups** of clauses of size $b$: $C_1, \ldots, C_{\frac{m}{b}}$
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To each group of variables, associate \(b\) colors corresponding to all possible assignments.
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\[
\begin{array}{ccc}
V_1 & V_i & V_{n/\log b} \\
\bigcirc & \bigcirc & \bigcirc \\
C_1 & C_j & C_{m/b}
\end{array}
\]
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\( \iff \) if a variable of \( V_i \) belongs to a clause of \( C_j \)
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\[ b - 1 \text{ colours out of the associated } b \]

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\( b \) colours out of all that satisfy one of the clauses
$d$-detecting sets

Given a set $X$ and a (mysterious) weight function $\omega : X \rightarrow \{-d, -d + 1, \ldots, d - 1, d\}$,

**Minimum size** of a collection $(S_1, \ldots, S_p)$ s.t.
if $\sum_{a \in S_i} \omega(a) = 0$ for every $i$ then $\omega \equiv 0$?
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$\Rightarrow$ encodes all subsets of $X$
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$\leq$ encodes all subsets of $X \Rightarrow p \geq \frac{|X|}{\log |X|}$.

$O\left(\frac{|X|}{\log |X|}\right)$ is **enough**! (Lindström ’65)
Conclusion
Thanks!
Homomorphism

**Definition**

*A graph $G$ is homomorphic to a graph $H$ if there is a function $f : V(G) \rightarrow V(H)$ that preserves adjacency.*
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$k$-coloring: homomorphism to $K_k$. 

Theorem (Hell, Neˇ setril '90)

For fixed $H$, “is $G$ homomorphic to $H$?” is NP-hard unless $H$ is bipartite.

Theorem (Cygan et al '16)

“is $G$ homomorphic to $H$?” cannot be solved in $O^{*}(|V(H)|^{\alpha} \cdot |V(G)|)$ time unless ETH fails.
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(graph on vertex set $\{\{1, \ldots, a\}^b\}$ with edges between disjoint sets).
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