Tight lower bounds for the complexity of multicoloring

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October 18th, 2016

Joint work with
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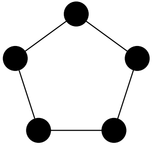
Coloring



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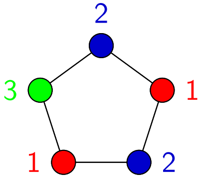

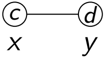

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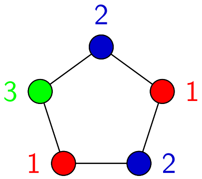

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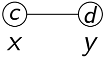

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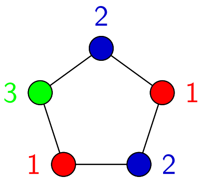

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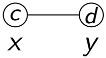

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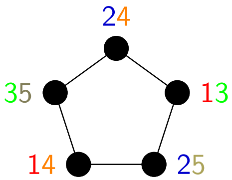

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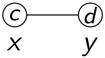

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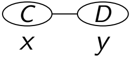

$$\Rightarrow \begin{cases} |C| = b \\ |D| = b \\ C \cap D = \emptyset \end{cases}$$

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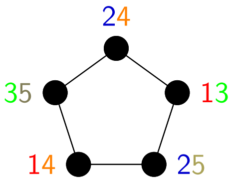

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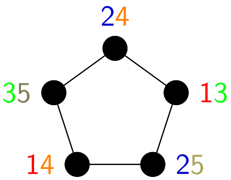

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$$lim_{b \to \infty} \frac{\chi_b}{b} = \chi_f$$

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- $a \geq 2b + 1$: NP-hard (Hell, Nešetril '90)

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For any $k \geq 3$, there is $\alpha > 0$ such that k-Coloring cannot be solved in $\mathcal{O}^*(2^{\alpha \cdot n})$ time unless ETH fails.

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Theorem (Björklund, Husfeldt '06)

k-Coloring can be solved in $\mathcal{O}^*(2^n)$ time.

Our result

Theorem (Nederlof '08)

a:b-Coloring can be solved in $\mathcal{O}^*((b+1)^n)$ time.

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Theorem (B., Kowalik, Pilipczuk, Socała, Wrochna '16)

There is $\alpha > 0$ such that, for appropriate ranges of values, a:b-Coloring cannot be solved in $\mathcal{O}^*((b+1)^{\alpha \cdot n})$ time unless ETH fails.

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We can also relax a:b-coloring: every vertex is assigned

- an integer $\in \{1, \ldots, b\}$ (number of colors to receive) and
- a subset of $\{1, \ldots, a\}$ (colors it's allowed to take).

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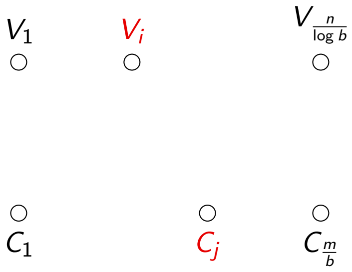

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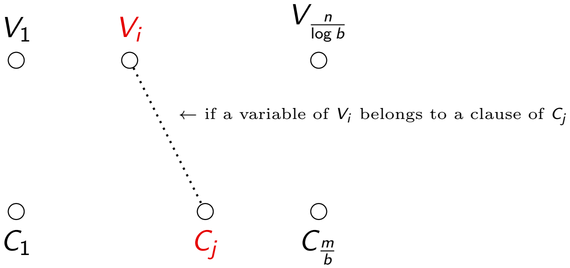

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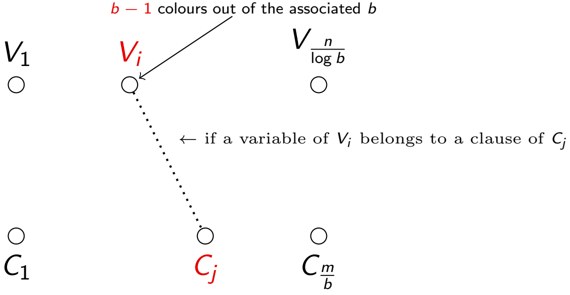

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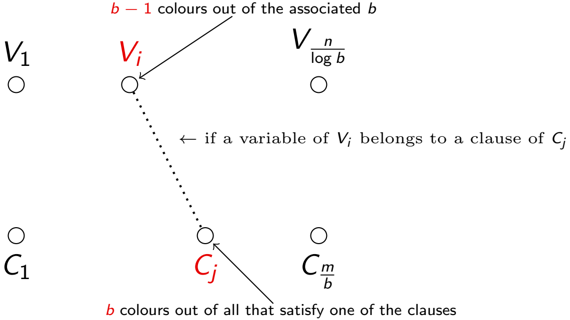

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Given a set X and a (mysterious) weight function $\omega : X \to \{-d, -d+1, \ldots, d-1, d\}$,

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$O(\frac{|X|}{\log|X|})$ is enough! (Lindström '65)

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