Edge-coloring Multigraphs

Daniel W. Cranston
Virginia Commonwealth University
dcranston@vcu.edu

joint with Landon Rabern

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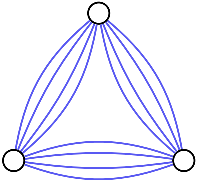

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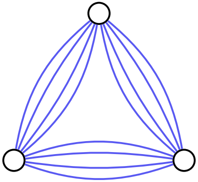

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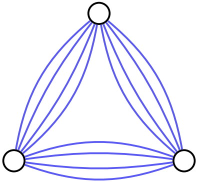

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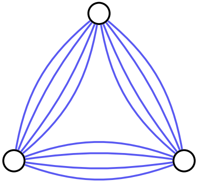

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Goldberg–Seymour Conj: Every multigraph G satisfies

$$\chi'(G) \leq \max\{\Delta(G) + 1, \mathcal{W}(G)\}.$$

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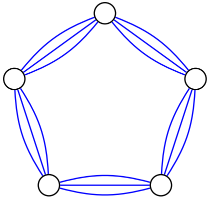

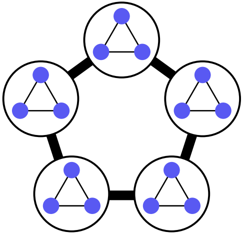

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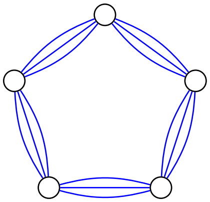

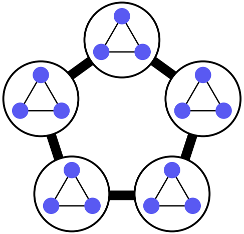

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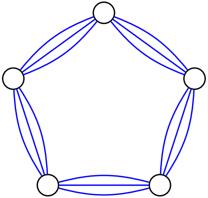

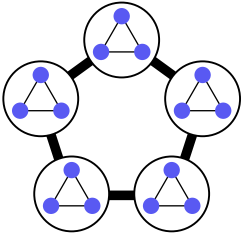

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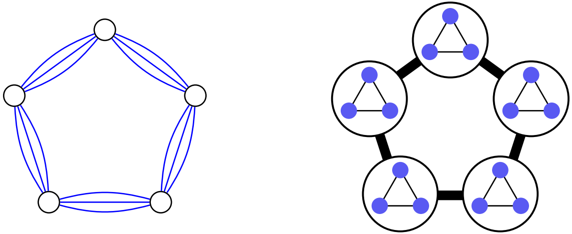

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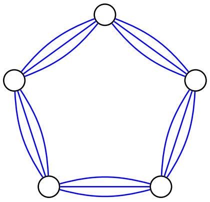

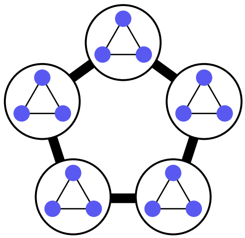

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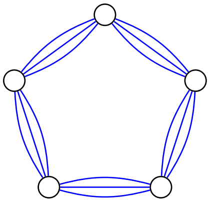

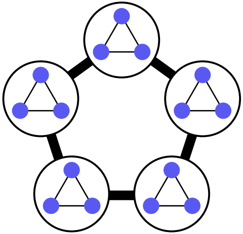

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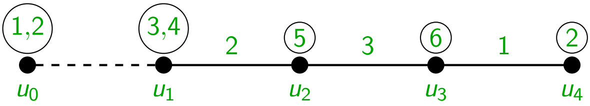

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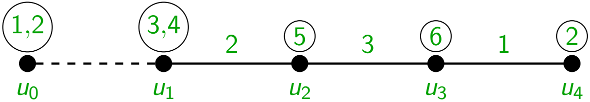

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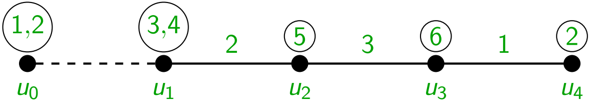

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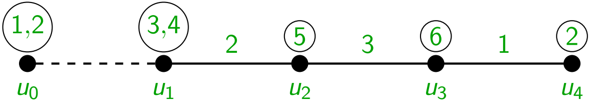

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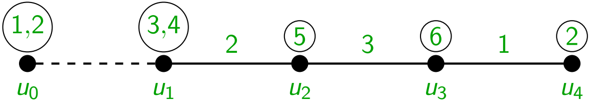

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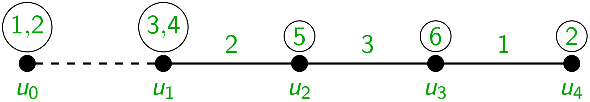

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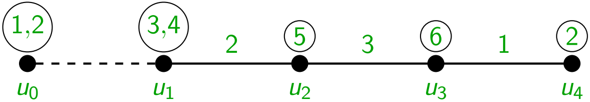

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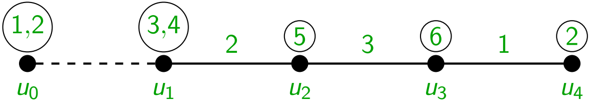

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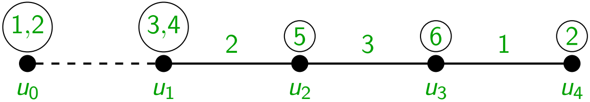

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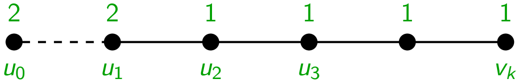

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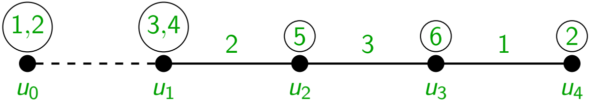

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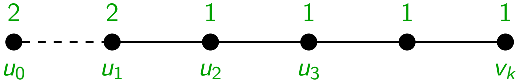

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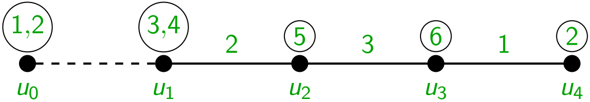

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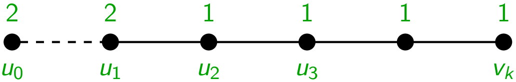

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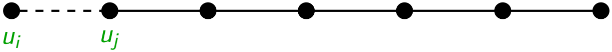

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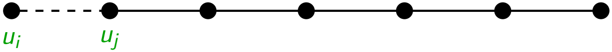

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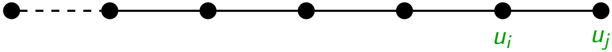

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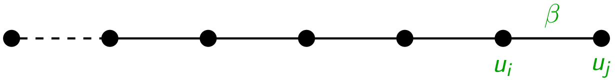

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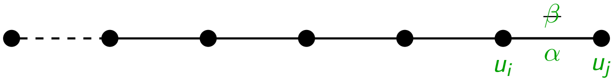

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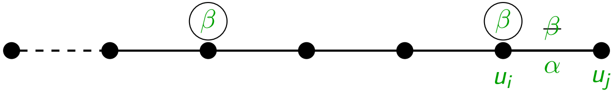

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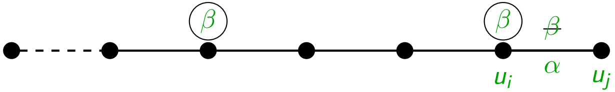

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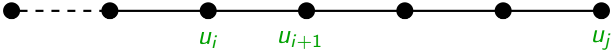

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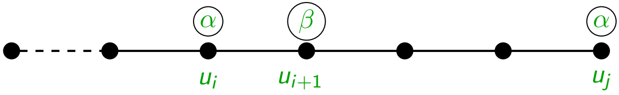

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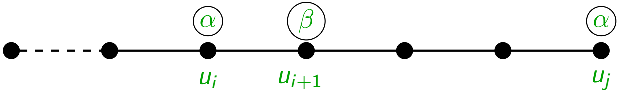

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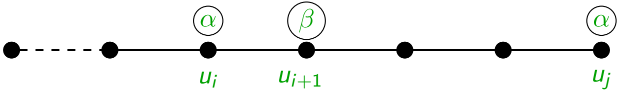

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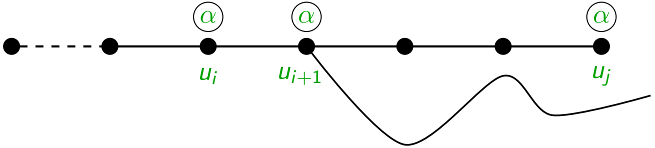

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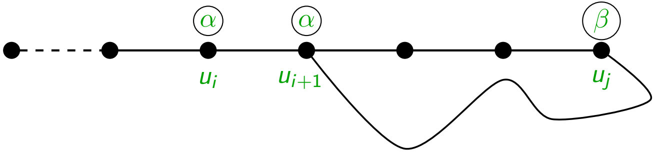

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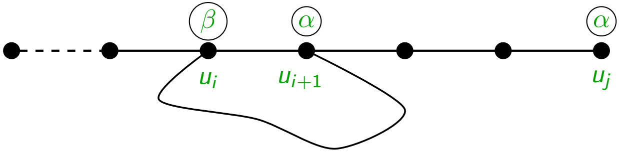

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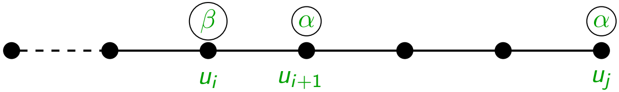

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In each case, win by induction hypothesis. ■

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Def: For a critical graph G with $\chi'(G) = k+1$, a vertex v is long if for some edge e incident to v and k-edge-coloring of $G - e$, some Vizing fan rooted at v has length at least 3; otherwise v is short.

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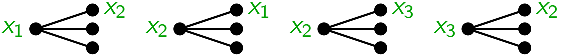

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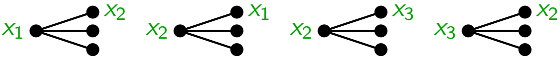

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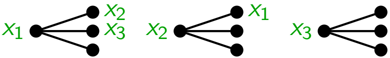

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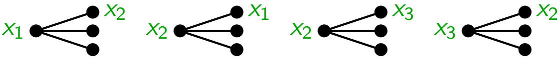

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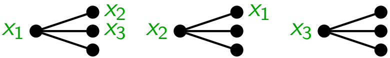

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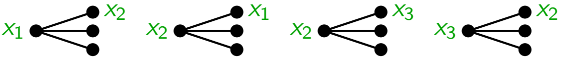

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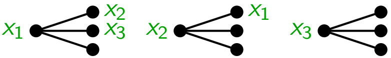

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Claim: Let F be Vizing fan at x w.r.t. k-edge-coloring of $G - xy$.
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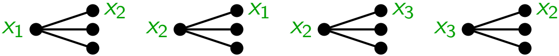

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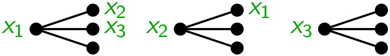

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Pf of Thm: (1) trivial; (2) reducible; (3, 4) $\sum_{i=1}^{3} d(x_i) < 2k$; (5) $\sum_{i=1}^{4} d(x_i) < 3k$; so (3)–(5) violate Tashkinov's Lemma. ■

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Parallel Edge Machine: Let φ be k-edge-coloring of $G - v_0v_1$. Choose $\alpha \in \overline{\varphi}(v_0)$ and $\beta \in \overline{\varphi}(v_1)$. Let $P = v_1 \cdots v_r$ be α, β-path with $e_i = v_iv_{i+1}$ for all $i \leq r - 1$. If v_i is short for all odd i, then for each $\tau \in \overline{\varphi}(v_0)$, we have a τ-colored $f_i = v_iv_{i+1}$ for each odd i.

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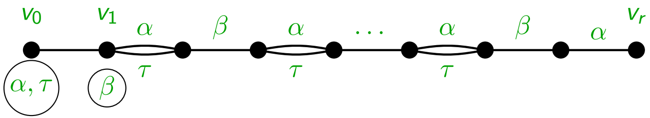

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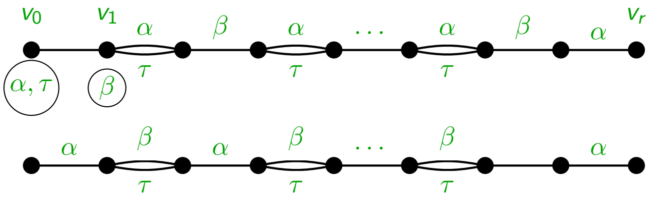

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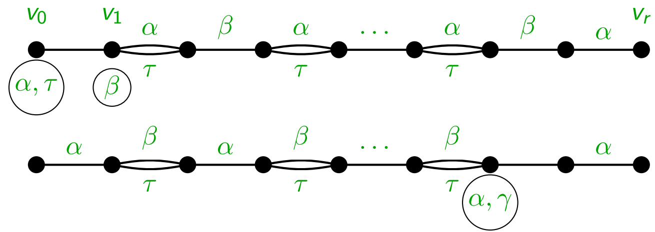

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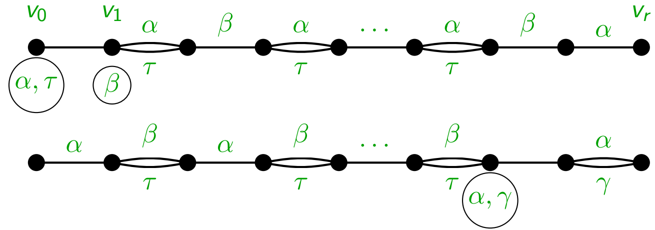
Parallel Edge Machine: Let φ be k-edge-coloring of $G - v_0 v_1$. Choose $\alpha \in \overline{\varphi}(v_0)$ and $\beta \in \overline{\varphi}(v_1)$. Let $P = v_1 \cdots v_r$ be α, β-path with $e_i = v_i v_{i+1}$ for all $i \leq r-1$. If v_i is short for all odd i, then for each $\tau \in \overline{\varphi}(v_0)$, we have a τ-colored $f_i = v_i v_{i+1}$ for each odd i.

Pf: Induction on r. Base case: v_1 is short. Induction step.



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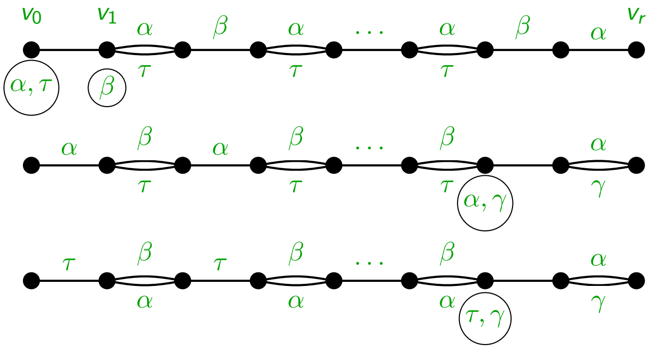

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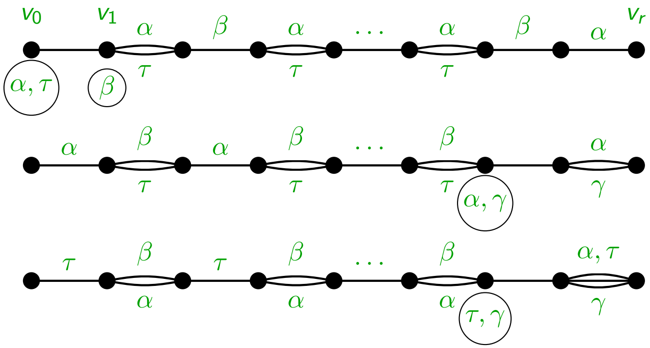

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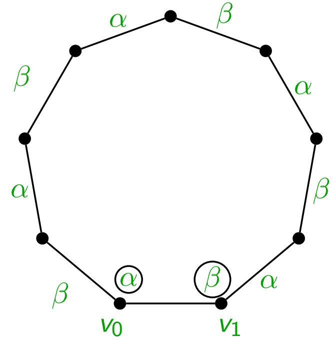

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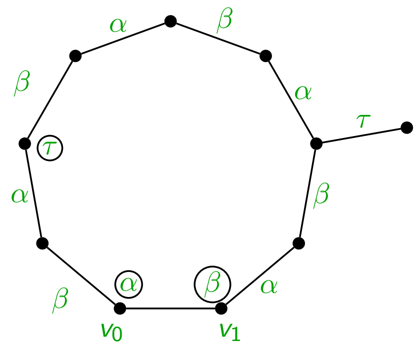

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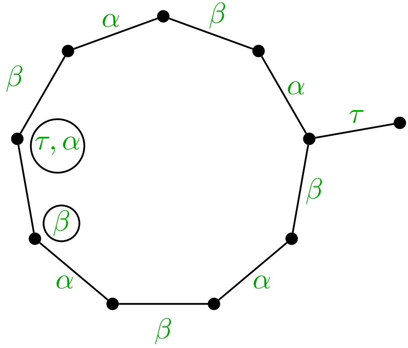

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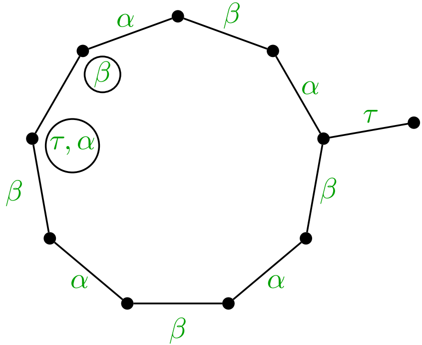


Overview Redux

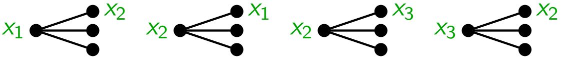
Thm: If Q is $L(G)$, then $\chi(Q) \leq \max\left\{\mathcal{W}(G), \Delta(G)+1, \frac{5\Delta(Q)+8}{6}\right\}$.

Key Lemma: If G is critical, then one of the following is true.

(1) G is elementary, i.e., $\chi'(G) = \mathcal{W}(G)$

(2) $\mu(G) > \frac{k}{2}$

(3) T has 3 long vertices x_1, x_2, x_3 s.t. these are Vizing fans



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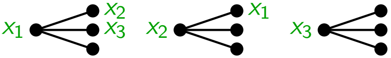

(5) T has 4 long vertices x_1, x_2, x_3, x_4

Claim: Let F be Vizing fan at x w.r.t. k-edge-coloring of $G - xy$.
If $S \subseteq V(F) - x$ and $|S| = 3$, then $d(x) < \frac{1}{4}\sum_{v \in S} d(v) \leq \frac{3}{4}\Delta(G)$.

Pf of Thm: (1) trivial; (2) reducible; (3, 4) $\sum_{i=1}^{3} d(x_i) < 2k$;
(5) $\sum_{i=1}^{4} d(x_i) < 3k$; so (3)–(5) violate Tashkinov's Lemma. ■