Propagation phenomena in nonlocal reaction-diffusion equations: An overview of the recent developments

Jérôme Coville

INRA Biostatistique et Processus Spatiaux (BioSP) Avignon - France

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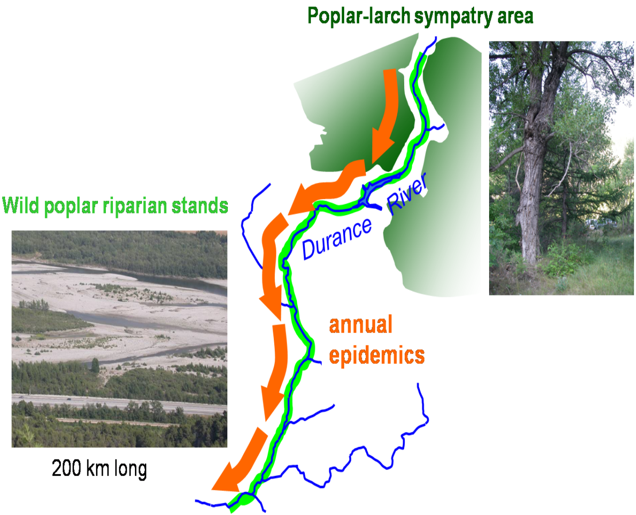
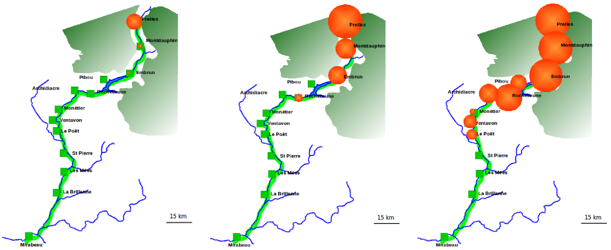
Outline of the Talk

Motivation

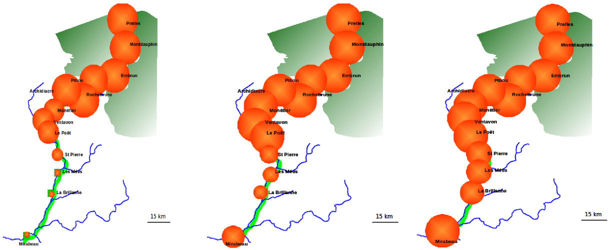
Study of the epidemic of the poplar rust in the Durance Valley



Epidemic Data: Halkett F.; Xhaard C. (INRA, UMR IAM)



Tournée 1 (du 9 au 11 juillet) | Tournée 3 (du 20 au 22 aout) | Tournée 4 (du 08 au 10 septembre)



Tournée 5 (du 30 septembre au 02 octobre) | Tournée 6 (du 20 au 24 octobre) | Tournée 7 (du 12 au 14 novembre)

Phenotypical & Genotypical Data: Halkett F.; Xhaard C.

Main Questions

1. ***How to explain the epidemic data ?***

2. ***How to explain the genotypical and phenotypical data ?***

3. ***Evaluate the gain of collecting genotypical/phenotypical data ?***

4. ***How to model interactions between genotypes/phenotypes and the space?***

A continuous mathematical approach

Reaction-dispersion models:
The population can be represented by a mean density $u(t,x)$ which is driven by:

$$\underbrace{\frac{\partial u}{\partial t}(t,x)}_{\text{Time variation}} = \underbrace{\mathcal{M}[u](t,x)}_{\text{Movement}} + \underbrace{f(t,x,u(t,x),\mathcal{K}[u](t,x))}_{\text{Growth}}.$$

Dispersion term $\mathcal{M}[u](t,x)$: Movement of the population (difference between individuals which come to location x and those which leave location x.)

Reaction term f: population growth rate. Depends on environmental characteristics (carrying capacity of the environment) and the structure of competition endures by the population $\mathcal{K}[u]$.

Dispersal models

Diffusion model: local dispersal into adjacent habitat

$$\mathcal{M}[u](t,x) = \partial_x^2 u(t,x);$$

$\Longrightarrow$ Local dispersal operator linked to random walks.

Integro-differential model: non-local dispersal

$$\mathcal{M}[u](t,x) = \int_{\mathbb{R}} J(|x-y|)\big(u(t,y) - u(t,x)\big)\,dy;$$

Dispersal kernel: $J(x-y)$ is the probability distribution of jumping from location y to location x.

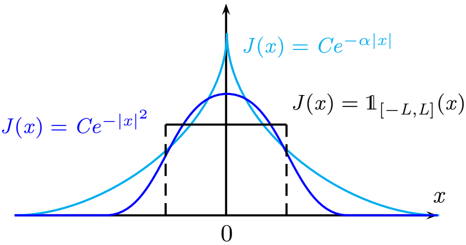
$\Longrightarrow$ Nonlocal dispersal operator. Can include long distance dispersal events.

Example of kernel dispersal

Thin-tailed kernel

Definition

$$\int_{\mathbb{R}} J(x)e^{\alpha x} < \infty \text{ for some } \alpha > 0.$$

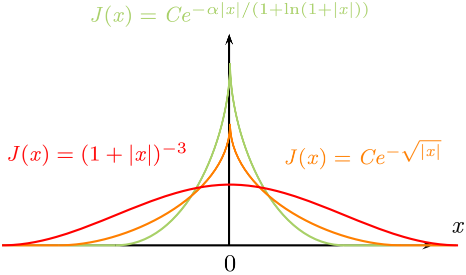


(Diekmann 1979, Thieme 1979, Schumacher 1980, Weinberger 1982, Coville et al. 2008)

Fat-tailed kernel

Definition

$$\text{for any } \alpha > 0,\ J(x) \geq e^{-\alpha|x|} \text{ for large } |x|.$$



(Medlock and Kot 2003, Yagisita 2009)

Modelling the growth

Population of size $N(t,x)$ at time t and position x. The increase in size:

$$N(t+\delta t,x) - N(t,x) = \text{(nb of birth-nb of death) during } \delta t.$$

Birth rate b and death rate d.

$$u(t+\delta t,x) - u(t,x) = bu(t,x)\delta t - du(t,x)\delta t.$$

$\rightsquigarrow \delta t \to 0$:

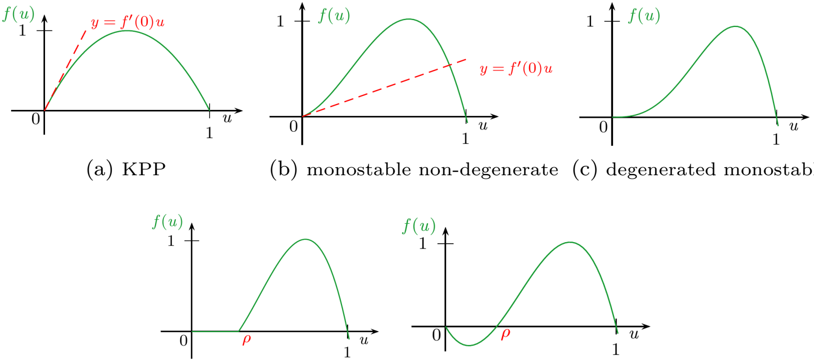
$$\boxed{\partial_t u(t,x) = (b-d)u(t,x).}$$

b and d encode the demographic processes considered

- Allee effect or Not
- Competition between the individuals or not

Example of population growth in homogeneous media

⇝ f is a function, $f(0) = f(1) = 0$ and $f'(1) < 0$.



(a) KPP (b) monostable non-degenerate (c) degenerated monostable

(d) ignition (e) bistable

Answer to the first question

The best model (*C, Fabre, Halkett, Soubeyrand, Xhaard, preprint*)

After a statistical treatment of the data within the class of Fisher-KPP demographic function, the best model that fits the epidemic data is :

$$\partial_t u(t,x) = J \star u(t,x) - u(t,x) + u(t,x)(r(x) - u(t,x)) \quad (1)$$

with

$$J(z) \sim e^{-|z|^\gamma} \quad \text{with} \quad \gamma = 0.15$$

and

$$r(x) := \begin{cases} K_1 \text{ in } & (R_0, +\infty) \\ K_2 \text{ in } & (-R_0, R_0) \\ K_3 \text{ in } & (-\infty, -R_0) \end{cases}$$

A first conclusion

Reaction diffusion equation with nonlocal dispersal are useful to explain real data !!!

To go further in the understanding of the data

We need to investigate in more details

- The propagation phenomena for solution of the equation (1)
- The role of the nonlinearity in such propagation phenomena
- ways to evaluate the speed of propagation
- the role of the heterogeneity
- the role of the kernel

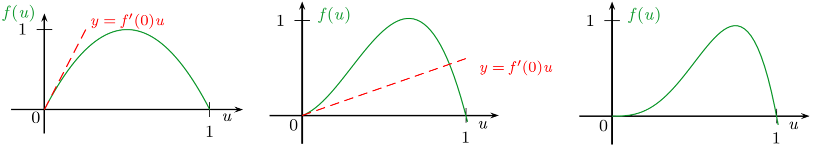
The Homogeneous Situation

The non local Reaction diffusion equation

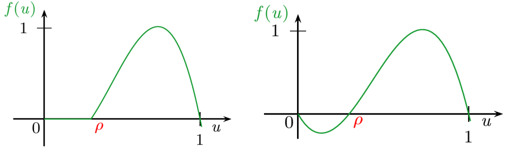
$$\partial_t u = J * u - u + f(u). \qquad (2)$$

⇝ $J \geq 0, \int_{\mathbb{R}} J(z)\,dz = 1$ and $\int_{\mathbb{R}} J(z)|z|\,dz < +\infty$.

⇝ f is a smooth function, $f(0) = f(1) = 0$ and $f'(1) < 0$.



(f) KPP (g) monostable non-degenerate (h) degenerated monostable



The main questions to ask

Initial data is front-like: $0 \leq u_0 \leq 1$, $\liminf_{x\to-\infty} u_0(x) > 0$,

- How does the stable state 1 invade or not the state 0?

$\rightsquigarrow$ Is there travelling fronts solution to (2)?

$\rightsquigarrow$ On what conditions on J and f such a solution exists

$\rightsquigarrow$ Can we characterise the spreading speed?

$\rightsquigarrow$ When such solution do not exist, what happens for the Cauchy problem ?

Known Results

Travelling wave: (*Bates-Fife-Ren-Wang (1997), Chen(1997), Alberti-Belletini(1998), Coville(2003,2006,2007), Yagisita (2009)*)

Assume J is as above and f is a bistable or ignition nonlinearity, then there exists a unique c such that there exists a *travelling wave* with speed c, i.e. there exists (φ, c) such that $\varphi \in L^\infty$, monotone and satisfying

$$\int_{\mathbb{R}} J(x-y)(\varphi(y)-\varphi(x))\,dy + c\varphi'(y) + f(\varphi(y)) = 0, \quad \text{in } \mathbb{R}$$

$$\varphi(-\infty) = 1 \quad \text{and} \quad \varphi(+\infty) = 0.$$

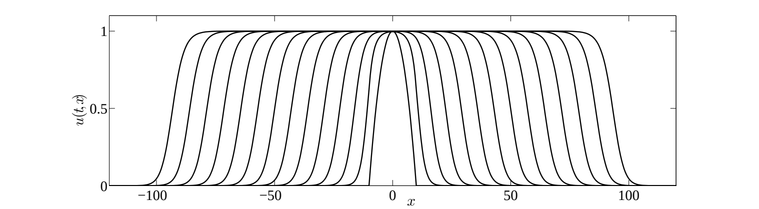
Known Results

Travelling wave: (*Schumacher 1980, Carr-Chmaj 2004, Coville et al. 2003,2007,2008, Zhao et al. (2007,2008,2009), Yagisita 2009, ...*)

Assume J is a thin tailed kernel and f is a monostable nonlinearity, then there exists c^* such that for all $c \geq c^*$, there exists a *travelling wave* with speed c, i.e. there exists (φ, c) such that $\varphi \in L^\infty$, monotone and satisfying

$$\int_{\mathbb{R}} J(x-y)(\varphi(y) - \varphi(x))\,dy + c\varphi'(y) + f(\varphi(y)) = 0, \quad \text{in } \mathbb{R}$$
$$\varphi(-\infty) = 1 \quad \text{and} \quad \varphi(+\infty) = 0.$$

(*Lutcher et al., 2005*): *if u_0 is compactly supported, the spreading speed c of u satisfies $c = c^*$, the minimal speed of traveling fronts.*



Numerical obs: the solution converges to a traveling front with constant profile.

Known Results

Characterisation of the spreading speed: (*Weinberger (1982,...), Zhao (2007,2008,...), Yagisita (2009)*)

Let f be a bistable, ignition or monostable nonlinearity and assume that J is such that the assumptions a front exists. Then

(i) Variational formulas for KPP type nonlinearities:

$$c^* = \inf_{\lambda \in \mathbb{R}} \frac{1}{\lambda} \left(\int_{\mathbb{R}} J(-z) e^{\lambda z}\, dz - 1 + f'(0) \right),$$

(ii) Linear vs nonlinear determinacy of the minimal speed,

(iii) The spreading speed is the minimal speed for the existence of travelling wave.

Known Results

Acceleration: (*Kot-Medlock (2003), Zhao (2007), Yagisita (2009), Garnier (2011)*)

Let f be a monostable nonlinearity such that $f'(0) > 0$ and assume that J a symmetric fat tailed kernel. Then

(i) $c^* = +\infty$, and for all $c > 0$, $\min_{|x|<ct} u(t,x) \to 1$ as $t \to \infty$

(ii) There exists $\rho > f'(0)$ such that $\forall \varepsilon \in (0, f'(0)), \lambda \in (0,1)$, there exists $T_{\varepsilon,\lambda}$ such that for $t \geq T_{\lambda,\varepsilon}$,

$$\min\left(J^{-1}\left(e^{-(f'(0)-\varepsilon)t}\right) \cap \mathbb{R}^+\right) \leq x_\lambda^\pm(t) \leq \max\left(J^{-1}\left(e^{-\rho t}\right) \cap \mathbb{R}^+\right)$$

where $x_\lambda(t)^+ := \sup\{x \,|\, u(t,x) = \lambda\}$ and $x_\lambda(t)^- := \inf\{x \,|\, u(t,x) = \lambda\}$.

Remarks :

Position of level sets accelerate like $J^{-1}\left(e^{-\gamma t}\right)$

- $J(z) \sim \frac{1}{|z|^\alpha}$, $\quad \rightsquigarrow C_0 e^{\frac{(f'(0)-\varepsilon)}{\alpha} t} \leq |x_\lambda^\pm(t)| \leq C_1 e^{\frac{\rho}{\alpha} t}$.
- $J(z) \sim e^{-|z|^\alpha}$, with $\alpha < 1$, $\quad \rightsquigarrow C_0 t^{\frac{1}{\alpha}} \leq |x_\lambda^\pm(t)| \leq C_1 t^{\frac{1}{\alpha}}$.

A Missing piece in the puzzle

Remaining open cases

$\rightsquigarrow$ Existence/Non existence of Travelling wave for fat tailed kernel J and a monostable f with some degeneracy at 0 (i.e. $\lim_{s\to 0} \frac{f(s)}{s^\beta} = c_\beta$) for some $\beta > 1$)

Acceleration or not?

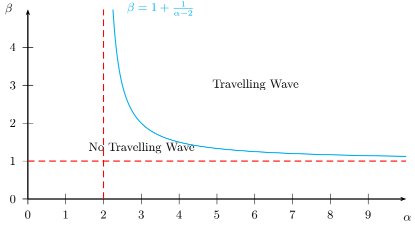
$\rightsquigarrow$ Characterisation of the Acceleration when it occurs when f is degenerated.

Existence/Non-existence of travelling wave

Theorem 1 (*Alfaro-C. (2016)*)

Let $f \in C^1_{loc}(\mathbb{R})$ be a monostable function such that $\lim_{s \to 0+} \frac{f(s)}{s^\beta} \leq C_\beta$ for some $C_\beta > 0$ and let J such that for $\alpha > 2$, $J(z) \sim \frac{1}{z^\alpha}$ as $z \to +\infty$. Then there exists c^* such that for all $c \geq c^*$, there exists a monotone *travelling wave* with speed c, iff

$$\beta - 1 \geq \frac{1}{\alpha - 2}.$$



Acceleration

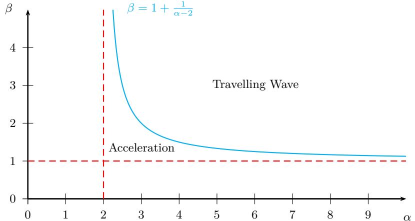
Theorem 2 (*Alfaro-C. (2016)*)

Let $f \in C^1_{loc}(\mathbb{R})$ and J as in the above Theorem and let $u(t,x)$ be the solution of the Cauchy problem with a front like initial data u_0. Assume

$$\beta - 1 < \frac{1}{\alpha - 2}.$$

Then $c^* = +\infty$ and $\lim_{t\to\infty} \frac{x_\lambda(t)}{t} = +\infty$. Moreover, there exists C_0 such that

$$x_\lambda(t) \leq C_0 t^{\frac{\beta}{(\alpha-1)(\beta-1)}}.$$



Estimates on the position of the Level set

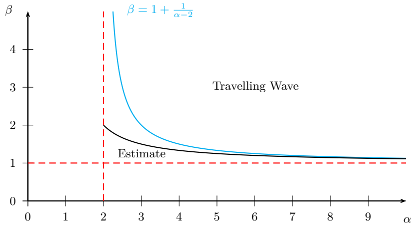
Theorem 3 (*Alfaro-C. (2016)*)

Let $f \in C^1_{loc}(\mathbb{R})$ and J as in the above Theorems and let $u(t,x)$ be the solution of the Cauchy problem with a front like initial data u_0.
Assume

$$\beta - 1 < \frac{1}{\alpha - 1}.$$

Then there exists $C_1 < C_0$ such that for t large

$$C_1 t^{\frac{1}{(\alpha-1)(\beta-1)}} \leq x_\lambda(t) \leq C_0 t^{\frac{1}{(\alpha-1)(\beta-1)} + \frac{1}{\alpha-1}}.$$



Remarks

Note that $\frac{1}{(\alpha-1)(\beta-1)} + \frac{1}{\alpha-1} < \frac{1}{(\alpha-2)(\beta-1)}$

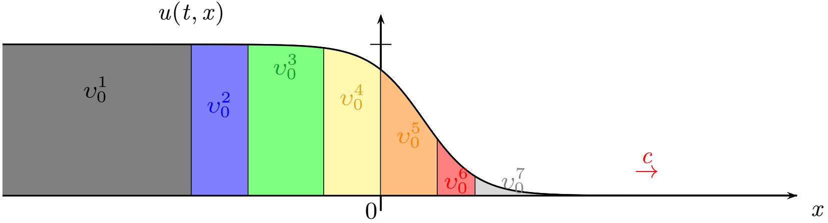
Note that $\frac{1}{(\alpha-1)(\beta-1)} + \frac{1}{\alpha-1} \to \frac{1}{(\alpha-1)(\beta-1)}$ as $\alpha \to \infty$

The existence of fronts still persists for algebraic kernel, kernel for which an exponential acceleration is observed when $f'(0) > 0$.

Inside Dynamics

Inside dynamics of an expanding solution

The expanding population u is made of several ***neutral* fraction** υ^k:



A coupled system of equations

At $t = 0$: $u_0(x) := u(0,x) = \sum_{k=1}^{7} \upsilon_0^k(x)$, with $\upsilon_0^k \geq 0$ for all $k \in \{1, \cdots, 7\}$.

The fractions υ^k only differ by their ***position*** and their ***allele*** (or their label):

$$\begin{cases} \partial_t \upsilon^k = \int_{\mathbb{R}} J(x-y)(\upsilon^k(t,y) - \upsilon^k(t,x))dy + \upsilon^k g\big(u(t,x)\big), \; t > 0, \; x \in \mathbb{R}, \\ \upsilon^k(0,x) = \upsilon_0^k(x), \; x \in \mathbb{R}. \end{cases} \quad (3)$$

$g(u) = f(u)/u$: the per capita growth rate of each fraction and of the total population u.

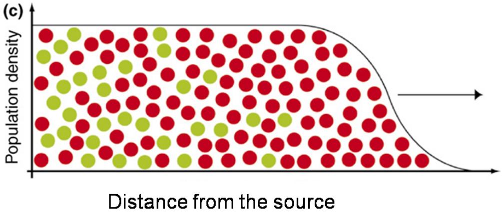
The characterization of the inside dynamics of a solution u is made through to the generic properties of its fractions.

Remark:

u and v^i, satisfy (3).
The uniqueness of the solution of the Cauchy problem $\Longrightarrow$ $u(x,t) = \sum_{i \in I} v^i(t,x)$ for all $t > 0$ and $x \in \Omega$.

Remark:

This Idea of looking at a composite population find its roots in the works of Vlad et al. 2004 and Hallatschek and Nelson 2008) on the notion of gene surfing. The above mathematical formulation is due to Roques, Garnier and their collaborators in 2012.



The integro-differential travelling waves

Thin-tailed kernel: (*Bonnefon, C, Garnier, Roques*)

• KPP case: If the initial density v_0 of the fraction decreases faster than U as $x \to +\infty$, then

$$\max_{x\in[A,+\infty)} v(t, x+ct) \to 0 \quad \text{as } t \to +\infty, \quad \text{for all } A \in \mathbb{R}.$$

• Weak Allee case with large speed: If the speed of the traveing wave is large $c \geq c^{**} > c^*$ then any initial density v_0 of the fraction decreasing faster than U as $x \to +\infty$, satisfies

$$\max_{x\in[A,+\infty)} v(t, x+ct) \to 0 \quad \text{as } t \to +\infty, \quad \text{for all } A \in \mathbb{R}.$$

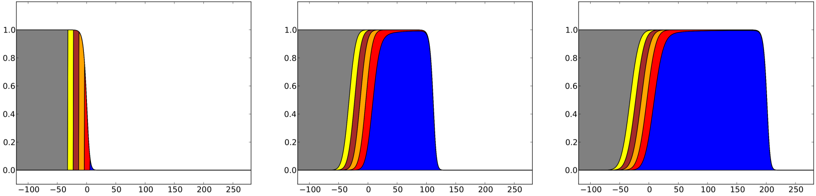
Remark:

$\rightsquigarrow$ Integro-differential travelling waves have the same inside dynamics as pulled monostable waves;

Consequences in population genetics

Thin-tailed kernel without Allee effect

Dynamics of the fraction v^k with $J(x) := J_{exp}(x) := \frac{1}{2}\, e^{-|x|}$.



Initial structure — at time $t = 40$ — at time $t = 72$

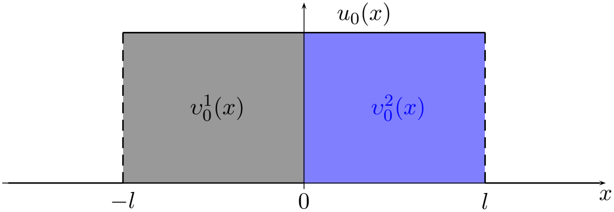
- Thin-tailed kernel without Allee effect $\rightsquigarrow$ get a spatial advantage at the forefront .
- Integro-differential travelling waves $\rightsquigarrow$ pulled travelling waves;

Inside dynamics of accelerating solutions

Dispersal kernel: let $\beta > 0$,

$$J(x) = \frac{\beta}{\pi(\beta^2 + x^2)} \text{ for all } x \in \mathbb{R}$$

Initial data u_0: let $l > 0$,



Fat-tailed kernel: (*Bonnefon, C, Garnier, Roques*)

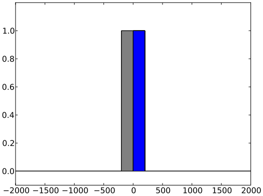
There exists a time $\tau > 0$ and a constant $\alpha > 0$ such that

$$\frac{\upsilon^1(t,x)}{u(t,x)} \geq \alpha \text{ for all } t \geq \tau \text{ and } x \in \mathbb{R}.$$

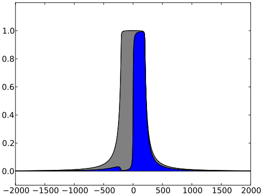
Consequences in population genetics

Thin-tailed kernel vs Fat-tailed kernel

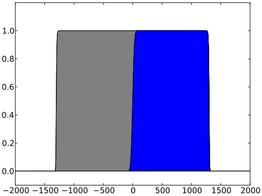
Dynamics of the fraction υ^1 and υ^2.



Initial structure

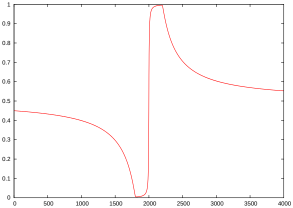


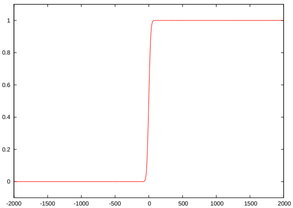
Fat-tailed kernel

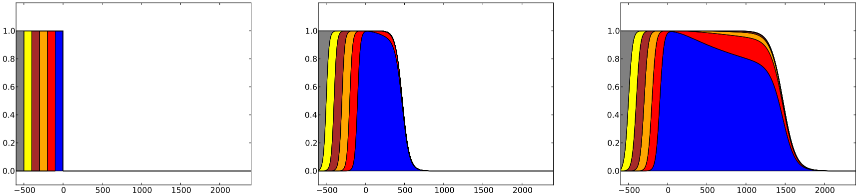


Thin-tailed kernel

Proportion of υ^2







(k) $t = 0$ (l) $t = 20$ (m) $t = 36$

Figure : Inside dynamics of the positive solution of the Cauchy problem with a kernel $J_{sqrt} = e^{-\sqrt{1+|x|}}/\sqrt{1+|x|}, f(s) = s(1-s)$ and a Heaviside type initial data i.e. $u_0 = \mathbb{1}_{(-\infty,0]}$.

The Heterogeneous Situation

The non local Reaction diffusion equation

$$\partial_t u = J * u - u + f(t, x, u). \tag{4}$$

$\rightsquigarrow$ $J \geq 0, \int_{\mathbb{R}} J(z)\,dz = 1$ and $\int_{\mathbb{R}} J(z)|z|\,dz < +\infty$.

$\rightsquigarrow$ $f(t, x, s)$ is a smooth function, $f(t, x, 0) = 0$.

New Problems emerged

- Notion of fronts ?
- Notion of bistability ?
- The notion of stability of the equilibria ?
- Shape of the propagation structure ?
- ”competition”
- .....

Some answers in a periodic setting

Several type of periodicity

- time periodic problem
- space periodic problem
- time and space periodic problems

Time periodic problem

- $J \in C^1(\mathbb{R}), J \geq 0, J(0) > 0, \int J = 1$
- $f(t,x,s) = f(t,s)$ with $f(t+T,s) = f(t,s)$ for all t, s.

Periodic bistable fronts: Bates-Chen (1999), Fang-Zhao (2015)

Assume J and f is bistable, then there exists a unique c such that there exists a time periodic function $\varphi(t, x-ct)$ solution of

$$\partial_t \varphi(t,z) - c\partial_z \varphi(t,z) = J \star \varphi(t,z) - \varphi(t,z) + f(t, \varphi(t,z))$$

- Bates-Chen $\rightsquigarrow$ continuity type method.
- Fang-Zhao $\rightsquigarrow$ Monotone semi-flow construction

Periodic monostable fronts: Liang-Zhao (2009)

Assume J is thin tailed and f is monostable,
$\rightsquigarrow$ Monotone semi-flow theory of Zhao et ale should give existence and characterisation of the spreading speed and of periodic fronts.

SPACE PERIODIC PROBLEM

- $J \in C^1(\mathbb{R})$, $J \geq 0$, $J(0) > 0$, $\int J = 1$
- f smooth, $f(t,x,s) = f(x,s)$ with $f(x+L,s) = f(x,s)$ for all x, s.

The notion of planar fronts make no sense in this situation.
$\rightsquigarrow$ New notion of front: Pulsating fronts, $u(t,x) := \varphi(x \cdot e + ct, x)$ such that $\varphi(s,.)$ is a periodic function for all s.

Spectral problems appears !!!
The problem :

$$J \star \psi - \psi + a(x)\psi = -\mu\psi$$

may be ill posed in the set $C(\mathbb{R}^N)$.

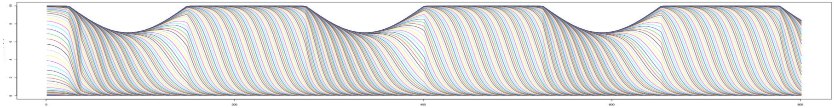


Figure : Simulation of pulsating front moving front the left to the right

SPACE PERIODIC PROBLEM

Pulsating front: Coville-Davila-Martinez (2012), Shen-Zhang (2012), Zhao et ale ...

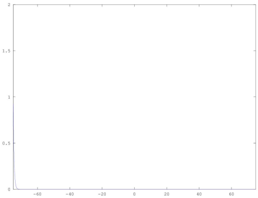
Assume J is thin tailed and f is of KPP type. Assume further that for all λ the linearised problem

$$\int_{\mathbb{R}^N} J(x-y)e^{\lambda(y-x).e}\psi(y)\,dy - \psi + f_u(x,0)\psi = -\mu_p(\lambda)\psi \quad \text{in } \mathbb{R}^N.$$

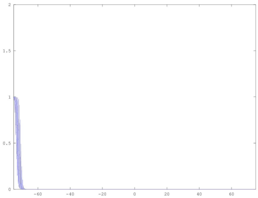
admit a solution (μ_p, ψ) with $\psi \in C(\mathbb{R}^N)$. Then there exists c^* such that for all $c \geq c^*$ there exists a function $\varphi(s,x)$ solution of

$$\begin{cases} c\varphi_s = \int_{\mathbb{R}^N} J(x-y)\varphi(s+(y-x)\cdot e, y)\,dy - \varphi + f(x,\varphi) \quad \forall s \in \mathbb{R},\ x \in \mathbb{R}^N \\ \varphi(s, x+k) = \varphi(s,x) \quad \forall s \in \mathbb{R},\ x \in \mathbb{R}^N,\ k \in \mathbb{Z}^N, \\ \lim_{s\to-\infty} \varphi(s,x) = 0 \quad \text{uniformly in } x, \quad \lim_{s\to\infty} \varphi(s,x) = p(x) \quad \text{uniformly in } x, \end{cases}$$

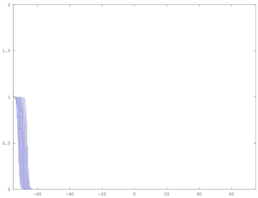
- Coville-Davila-Martinez $\rightsquigarrow$ PDE approach.
- Shen-Zhang $\rightsquigarrow$ dynamical system approach.
- Zhao $\rightsquigarrow$ Monotone semi-flow construction



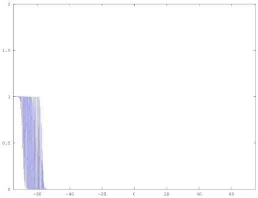
(a) $t = 0$



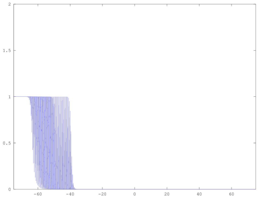
(b) $t = 10$



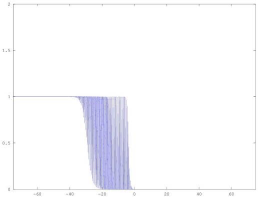
(c) $t = 20$



(d) $t = 40$



(e) $t = 80$



(f) $t = 160$

Figure : Simulation of the solution of the Cauchy problem when μ_0 is not associated to a eigenfunction function. J is a Gaussian,$f(x, s) = a(4x)s(1 - s)$ where $a(x) = 1 - \sqrt{|x|}$ on $[-1, 1]$ and is extended by periodicity. Pics are appearing at the forefront and propagates at a constant speed

Time and space periodic case

- $J \in C^1(\mathbb{R})$, $J \geq 0$, $J(0) > 0$, $\int J = 1$, J is thin tailed
- f smooth, $f(t,x,s)$ with $f(t+T, x+L, s) = f(t,x,s)$ for all t, x, s, f is a KPP type

As above the notion of planar fronts make no sense in this situation.
$\rightsquigarrow$ New notion of front:"periodic" Pulsating fronts, $u(t,x) := \varphi(x \cdot e + ct, t, x)$ such that for all s $\varphi(s,\cdot,\cdot)$ is a periodic function of t and x.

More spectral problems appears !!!
What is the right notion of eigenvalue for a problem :

$$\partial_t \psi - J \star \psi + \psi - a(t,x)\psi = \mu\psi.$$

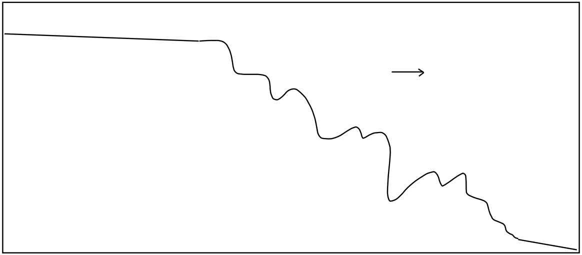
Pulsating wave and spreading speed: Shen-Rawal (2012), Zhao et ale ? ...

- Existence, stability of pulsating wave, existence of a minimal speed
- Characterisation of the Spreading speed

GENERIC NONLINEARITY

- $J \in C^1(\mathbb{R}), J \geq 0, J(0) > 0, \int J = 1,$ J is thin tailed
- f smooth, $f(t, x, s)$ with no a priori structure

As above the notion of fronts need a right definition.
⇝ New notion of front:"transition wave" defined by Berestycki and Hamel $u(t,x) := \varphi(t,x)$ a entire solution which looks like at $\pm$ to the stationary solution.



Some Results

- Lim and Zlatos, J compactly supported, f is a time and space "generic" KPP type nonlinearity: Existence of a transition front
- Shen-Shen, J is thin tailed, f is a "generic" time bistable or ignition nonlinearity: Existence, regularity and stability of a transition front
- Shen and Shen-Shen, J is thin tailed, f is a "generic" time kpp nonlinearity: Existence, regularity and stability of a transition front

Remark

There is no speed associated to these solution $\rightsquigarrow$ Notion of spreading speed to be adapted

Invasion along a environmental cline

Biological invasion by adaptation to local conditions

Generic Context

- ⇝ A species is usually well adapted to an optimal environment
- ⇝ Because of **Temperature, light, weather condition,** . . ., new territories are often not optimal for the growth of the invading population
  - ▶ To colonize this new territories and extend its repartition area, this local conditions force the population to have an "adaptation" process

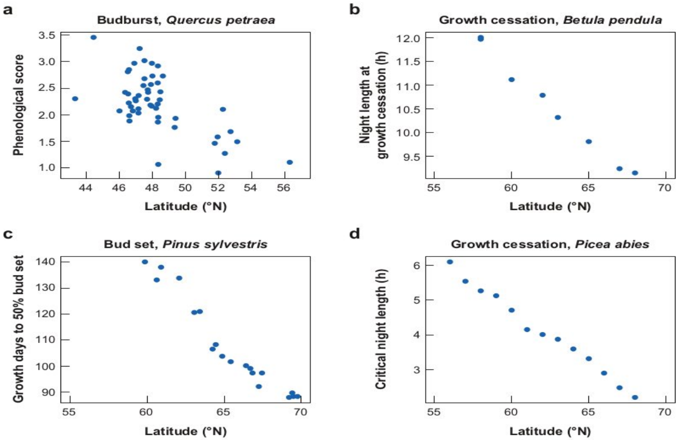
The case of an environmental cline

- ⇝ The environmental changes follow a **gradient** of temperature, light, antibiotic, . . .
- ⇝ The expansion of the species is possible through the **adaptation of one or more phenotypic traits**

$\Longrightarrow$ **Strong relation between the trait value and the position** .

Example of environmental cline

Studies of phenotypic trait related to budburst, budset and growth shows the presence of a latitudinal gradient ((a) Oak, (b) Birch, (c) Pine, (d) Spruce):



O. Savolainen, T. Pyhajarvi et T. Knurr, Annu. Rev. Ecol. Evol. Syst. 2007.

Clonal Population invading an environmental cline

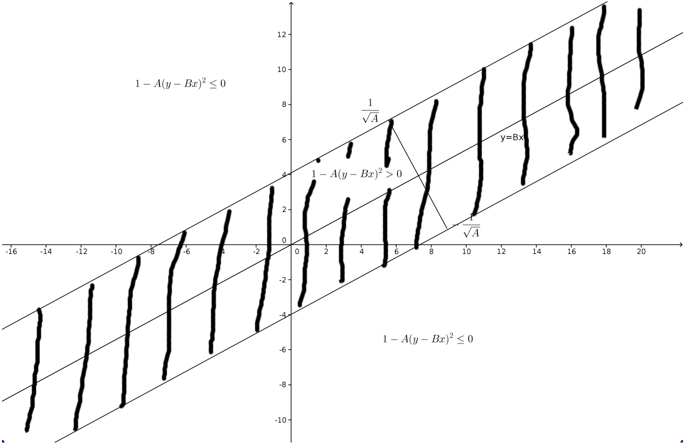
The Model considered

$n(t,x,y)$: density of population at time $t \geq 0$, position $x \in \mathbb{R}$, and with a phenotypic trait $y \in \mathbb{R}$:

$$\partial_t n(t,x,y) - \partial_{xx} n(t,x,y) - \partial_{yy} n(t,x,y) = \left(r_{max} - A(y - Bx)^2 - \frac{1}{K}\int_{\mathbb{R}} k(y,y')n(t,x,y')\, dy' \right) n(t,x,y)$$

A: Selection intensity
B: slope of the cline
K: carrying capacity .

Impact of A and B



Invasion

Characterisation of the invasion

$\rightsquigarrow$ Study of planar wave solutions of the equation: solutions which moves with a constant speed and constant shape.

Travelling wave solution

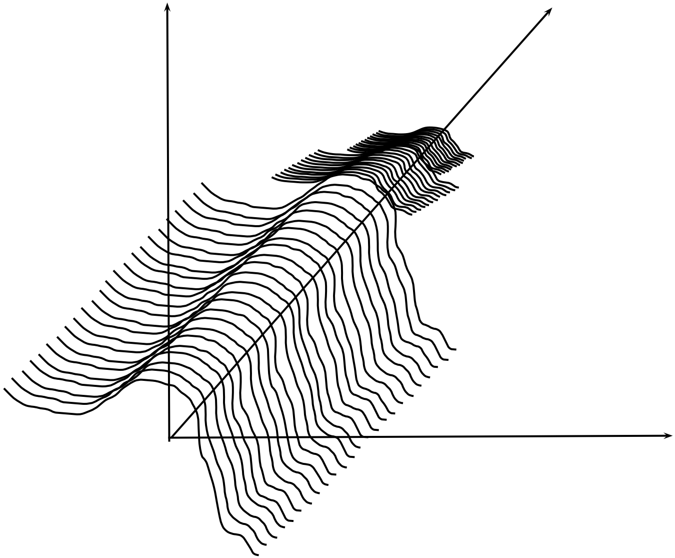
$\rightsquigarrow$ Solutions of the form $\psi(x + ct, z)$ where c is a speed to be determined and $\psi := \psi(\xi, z)$ is a profile satisfying

$$\partial_{\xi\xi}\psi - (B^2 + 1)\partial_{zz}\psi - 2B\partial_{\xi z}\psi - c\partial_{\xi}\psi = \left(1 - Az^2 - \int_{\mathbb{R}} k(z, z')\psi(\xi, z')\, dz'\right)\psi \quad \text{in}$$

with a *had hoc* behaviour as $x \to \pm\infty$.

Theorem (Alfaro-C-Raoul)

Assume $0 < k_{min} \leq k \leq k_{max}$. There exists a critical speed $c^* > 0$ such that, for all $c \geq c^*$, there exists (c, ψ) a positive solution of the TW problem. Moreover, no positive bounded front exists if $0 \leq c < c^*$.



Research IDEas

Research directions

Many research directions have emerged these last couple of years:

- Propagation phenomena for the nonlocal Fisher-KPP equation

$$\partial_t u(t,x) = \partial_{xx} u + u(1 - k \star u).$$

  Questions: What happens when ∂_{xx} is replaced by nonlocal diffusion
- Can space heterogeneity prevent acceleration ?
- Inside dynamics of accelerated solutions
- Invasion by adaptation process, sexual reproduction model?

Thanks you for your attention and good night