Universal properties of global equivariant Thom spectra

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Introduction

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- forget the $O(n)$-actions

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- similarly: $\pi_k^G(X)$ for $k \in \mathbb{Z}$

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The global stable homotopy category is

$$\mathcal{GH} \; = \; \mathrm{Sp}^O[\text{global equivalences}^{-1}] \; ,$$

the localization of orthogonal spectra at the class of global equivalences.

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$\Longrightarrow$ 'global functors' ('inflation functors')

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ko$(V) =$ finite configurations of points in S^V
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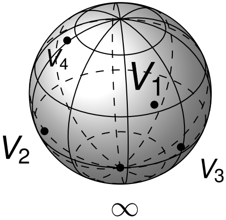

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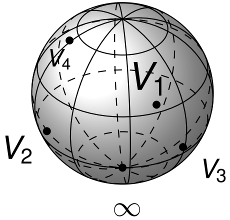

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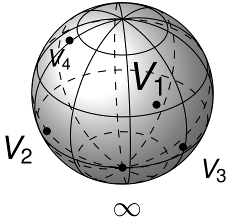

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$(H\mathbb{Z})(V) \;=\; Sp^\infty(S^V)$
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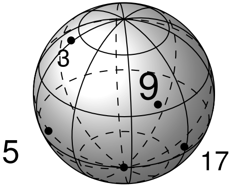

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- **MO** is ultra-commutative, **mO** is not.

The rank filtration of **mO**

$\mathbf{mO}_{(m)}$ is the orthogonal subspectrum of **mO** with

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$$\mathrm{Sp}^O \longrightarrow (\text{sets}) \;, \quad X \longmapsto X(\nu_m)^{O(m)} \;,$$

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Global homotopy type of $\mathbf{mO}_{(m)}$

Theorem

There is a global equivalence

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So

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Global morphisms out of **mO**

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The orthogonal spectrum $\mathbf{mO}_{(m)}$ *represents the functor*

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The inverse limit and derived limit are formed along

$$E_m^{O(m)}(S^{\nu_m}) \xrightarrow{\text{res}^{O(m)}_{O(m-1)}} E_m^{O(m-1)}(S^{\nu_{m-1}} \wedge S^1) \cong E_{m-1}^{O(m-1)}(S^{\nu_{m-1}})$$

and ‘ev’ is evaluation at the inverse Thom classes $\tau_{O(m),\nu_m}$.

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Example

The classes

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correspond to a global ring spectrum morphism $\mathbf{mU} \longrightarrow \mathbf{KU}$. Since $\mathbf{mU}$ is connective, this lifts to a morphism $\mathbf{mU} \longrightarrow \mathbf{ku}^c$ to global connective K-theory.

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$$\begin{aligned}(bR)_m^{O(m)}(S^{\nu_m}) &\cong [\![\mathbf{mO}_{(m)}, bR]\!] \\ &\cong [S^m \wedge BO^{-\gamma_m}, R] \cong R^{-m}(BO^{-\gamma_m})\end{aligned}$$

the inverse Thom class t_m corresponds to the Thom class of the virtual bundle $-\gamma_m$ over $BO(m)$.

Subqotients of the rank filtration

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$$\mathrm{Free}_{O(m-1),\nu_{m-1}\oplus\mathbb{R}} \longrightarrow \mathrm{Free}_{O(m),\nu_m} \longrightarrow \mathrm{Free}_{O(m),\nu_m} S^{\nu_m} \longrightarrow$$

which is isomorphic to

$$S^{-1} \wedge M_{\mathrm{gl}} O(m-1) \longrightarrow M_{\mathrm{gl}} O(m) \longrightarrow \Sigma^\infty_+ B_{\mathrm{gl}} O(m) \longrightarrow$$

Applying $S^m \wedge -$ gives the distinguished triangle

$$\mathbf{mO}_{(m-1)} \xrightarrow{\mathrm{incl}} \mathbf{mO}_{(m)} \longrightarrow S^m \wedge \Sigma^\infty_+ B_{\mathrm{gl}} O(m) \longrightarrow \qquad \square$$

Building **mO** by killing transfers

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mO *is built from* $\mathbb{S}$ *by killing* $\mathrm{Tr}_{O(m-1)}^{O(m)}$ *for* $m \geq 1$*.*

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- The morphism ∂ is classified by

$$\mathrm{Tr}_{O(m-1)}^{O(m)}\left(\tau_{O(m-1),\nu_{m-1}}\right) \quad \text{in } \pi_{m-1}^{O(m)}(\mathbf{mO}_{(m-1)}),$$

  the 'dimension shifting' transfer.

Global description of $\underline{\pi}_0(\mathbf{mO})$

Since $B_{\mathrm{gl}}G$ represents $\pi_0^G(-)$, the composite

$$\Sigma_+^\infty B_{\mathrm{gl}}O(m+1) \xrightarrow{\partial} S^{-m} \wedge \mathbf{mO}_{(m)} \xrightarrow{q} \Sigma_+^\infty B_{\mathrm{gl}}O(m)$$

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Corollary

The action of the Burnside ring global functor on the unit element $1 \in \pi_0^e(\mathbf{mO})$ *induces an isomorphism of global functors*

$$\mathbb{A}/\langle \mathrm{tr}_e^{O(1)} \rangle \cong \underline{\pi}_0(\mathbf{mO}) .$$

Summary

The fundamental relation $\mathrm{tr}_e^{O(1)}(1) = 0$ implies the more familiar

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Let G be a compact Lie group. An $\mathbb{F}_2$-basis of $\pi_0^G(\mathbf{mO})$ is given by the classes $\mathrm{tr}_H^G(1)$, indexed by conjugacy classes of closed subgroups H of G whose Weyl group is finite and of odd order.

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- Orthogonal spectra and global equivalences provide a convenient model
- The global perspective reveals universal properties of equivariant Thom spectra

Induction versus transfer

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Why is the TP-construction bijective only for $G \cong$finite$\times$torus?

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induction isomorphism:

$$\begin{array}{rcl} \mathcal{N}^H_{n-d}(X) & \xrightarrow{\mathrm{Ind}^G_H} & \mathcal{N}^G_n(G \times_H X) \\ [M, h] & \longmapsto & [G \times_H M, G \times_H h] \end{array}$$

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Answer:
Different formal behaviour of induction / transfer.
So no chance for an isomorphism in general.

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$p : S(V \oplus \mathbb{R}) \longrightarrow S^V$ stereographic projection
represents a tautological equivariant bordism class

$$d_{G,V} \in \widetilde{\mathcal{N}}^G_{|V|}(S^V)$$

Correction by tautological class

Recall: $L = T_H(G/H)$ tangent H-representation,
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Proposition

For every closed subgroup H of a compact Lie group G and every H-space X the following diagram commutes:

$$\begin{array}{ccc}
\mathcal{N}^H_{n-d}(X) & \xrightarrow{TP} & \mathbf{mO}^H_{n-d}(X_+) \\
 & & \downarrow d_{H,L} \times - \\
\mathrm{Ind}^G_H \downarrow \cong & & \mathbf{mO}^H_n(S^L \wedge X_+) \\
 & & \cong \downarrow \mathrm{Tr}^G_H \\
\mathcal{N}^G_n(G \times_H X) & \xrightarrow[TP]{} & \mathbf{mO}^G_n((G \times_H X)_+)
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- the tautological class $d_{H,L}$ measures the failure of Thom-Pontryagin map to commute with induction/transfer.

Stable equivariant bordism and **MO**

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Corollary (Bröcker-Hook '72)

After formally inverting all tautological classes in $\mathcal{N}_*^G(-)$ *and in* $\mathbf{mO}_*^G(-)$*, the Thom-Pontryagin construction becomes an isomorphism for all compact Lie groups* G *and all* G*-spaces* X*.*

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- tom Dieck's homotopical equivariant bordism:

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- Does $\mathbf{mO}_*^G(-)$ describe any geometric G-bordism theory? We need to twist induction by the tangent representation...
- Are there generalizations to equivariant bordism theories with more structure ($\mathbf{mSO}_*^G$, $\mathbf{mSpin}_*^G$, $\mathbf{mU}_*^G$, $\ldots$)? Induction needs extra structure on G/H ...

[back to main story]