A Class of Artin-Schreier Curves With Many Automorphisms

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Joint work with **Irene Bouw**, **Wei Ho**, **Beth Malmskog**,
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So Artin-Schreier extensions are the wild analogues of (tame cyclic) Kummer extensions $\mathbb{F}(x, y)/\mathbb{F}(x)$ where $\mu_n \subset \mathbb{F}$ and

$$y^n = F(x) \quad \text{with} \quad p \nmid n .$$

Our Main Protagonist

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The genus of C_R is $g(C_R) = \dfrac{\deg(R)(p-1)}{2}$.

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  (Case $p = 2$ in van der Geer & van der Vlugt, *Comp. Math.* **84**, 1992)
- Maximal over suitable fields and hence a good source for algebraic geometry codes.
- Other cool geometric and algebraic properties:
  - Very large and interesting automorphism group.
  - Supersingular family (Jacobian is isogenous to a product of supersingular elliptic curves).

Outline

For odd p, this is our protagonist's story:

1. Point counts
2. Zeta function (almost)
3. Automorphism group, including fields of definition
4. Zeta function
5. Examples

Two Magic Structures

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- W is an $\mathbb{F}_p$-vector space of dimension $2h$.
- We have a very explicit description of the elements of W (later).

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Proposition

Let

$$C_R : y^p - y = xR(x)$$

with $R(x) \in \mathbb{F}_q[x]$ *additive of degree* p^h. *Then for any extension* $\mathbb{F}_{p^n}$ *of* $\mathbb{F}_q$, *the number of* $\mathbb{F}_{p^n}$*-rational points is*

$$\#C_R(\mathbb{F}_{p^n}) = \begin{cases} p^n + 1 & \text{for } n \text{ odd,} \\ p^n + 1 \pm (p-1)p^{h+n/2} & \text{for } n \text{ even.} \end{cases}$$

Point Counts

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Corollary

C_R *is either maximal* $(+)$ *or minimal* $(-)$ *for* n *even.*

Idea of the Proof

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Total count: $\#C_R(\mathbb{F}_{p^n}) = p^{2h+1}N_n + 1$.

Zeta Function

The *zeta function* of a curve C of genus g over a finite field $\mathbb{F}_q$ is

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If we write $L_{C,q^n}(t) = \prod_{i=1}^{2g} (1 - \alpha_i t)$, then $\sum_{i=1}^{2g} \alpha_i = \#C(\mathbb{F}_{q^n}) - q^n - 1$.

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Applying this to C_R, we obtain for all i:

$$\alpha_i = \begin{cases} \pm q^n & \text{when } n \text{ is odd,} \\ \pm q^{n/2} & \text{when } n \text{ is even.} \end{cases}$$

L-Polynomial of C_R (Almost)

Proposition

Let $C_R : y^p - y = xR(x)$ with $R(x) \in \mathbb{F}_q[x]$ additive of degree p^h. Then for any extension $\mathbb{F}_{p^n}$ of $\mathbb{F}_q$, we have

$$L_{C_R,p^n}(t) = \begin{cases} (1 \pm p^n t^2)^g & \text{when } n \text{ is odd,} \\ (1 \pm p^{n/2} t)^{2g} & \text{when } n \text{ is even.} \end{cases}$$

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Since all the slopes of the Newton polygon of the L-polynomial are equal to $1/2$, we obtain:

Corollary

The Jacobian of C_R is isogenous to a product of supersingular elliptic curves. So C_R is supersingular.

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Unfortunately, the "$\pm$" is surprisingly hard to resolve.

Automorphism Group of C_R

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Proposition

Assume without loss of generality that $R(x)$ is monic.

- *If $R(x) = x$, then $\mathrm{Aut}(C_R) \cong SL_2(\mathbb{F}_p)$.*
- *If $R(x) = x^p$, then $\mathrm{Aut}(C_R) \cong PGU_3(\mathbb{F}_p)$ (Hermitian case).*
- *If $R(x) \notin \{x, x^p\}$, then every element of $\mathrm{Aut}(C_R)$ fixes ∞.*

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It therefore suffices to compute the group

$$\mathrm{Aut}^{\infty}(C_R)$$

of automorphisms that fix ∞.

The group $\mathrm{Aut}^\infty(C_R)$

We have the following commutative diagram:

$$\begin{tikzcd} C_R \arrow[r, "\varphi"] \arrow[d, "{(x,y) \mapsto x}"'] & C_R \arrow[d, "{(x,y) \mapsto x}"] \\ \mathbb{P}^1 \arrow[r, "\tilde{\varphi}"] & \mathbb{P}^1 \end{tikzcd}$$

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As a result, all automorphisms in $\mathrm{Aut}^{\infty}(C_R)$ have the form

$$\varphi(x, y) = (ax + c, dy + B(x))$$

with $a, c, d, B(x)$ live in some extension of $\mathbb{F}_p$.

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Structure of $\mathrm{Aut}^\infty(C_R)$: We have $\mathrm{Aut}^\infty(C_R) = P \rtimes H$ where

- H is a boring group of dilations.
- P is an interesting group of translations.

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H is cyclic, and its order can be easily determined from $R(x)$.

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where

- $c \in W$, or equivalently, there exists a unique additive polynomial $B_c(x) \in \mathbb{F}_q[x]$ such that

$$B_c(x)^p - B_c(x) = cR(x) + R(c)x \ .$$

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- Compute $L_{C_R/A,\mathbb{F}_{p^n}}(t)$ directly.

Resolving $\pm$ in $L_{C_R,p^n}(t)$

Theorem

Let $m = a_h$ if $h = 0$ and $m = m_M$ when $h > 0$.

If $p \equiv 1 \pmod 4$, then

$$L_{C_R,\mathbb{F}_{p^n}}(t) = \begin{cases} (1-p^n t^2)^g & \text{when } n \text{ is odd}, \\ (1-p^{n/2}t)^{2g} & \text{when } n \text{ is even and } m_M = \square \text{ in } \mathbb{F}_{p^n}, \\ (1+p^{n/2}t)^{2g} & \text{when } n \text{ is even and } m_M \neq \square \text{ in } \mathbb{F}_{p^n}. \end{cases}$$

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Some Examples

Examples with $h = 0$, i.e. $R(x) = mx$

The following two maximal curves are additions to the database www.manYPoints.org:

- The genus 5 curve $y^{11} - y = mx^2$, with m a nonsquare in $\mathbb{F}_{11^4}$, is maximal over $\mathbb{F}_{11^4}$.
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Families of examples with $h > 0$ and $R(x) = mx^{p^h}$

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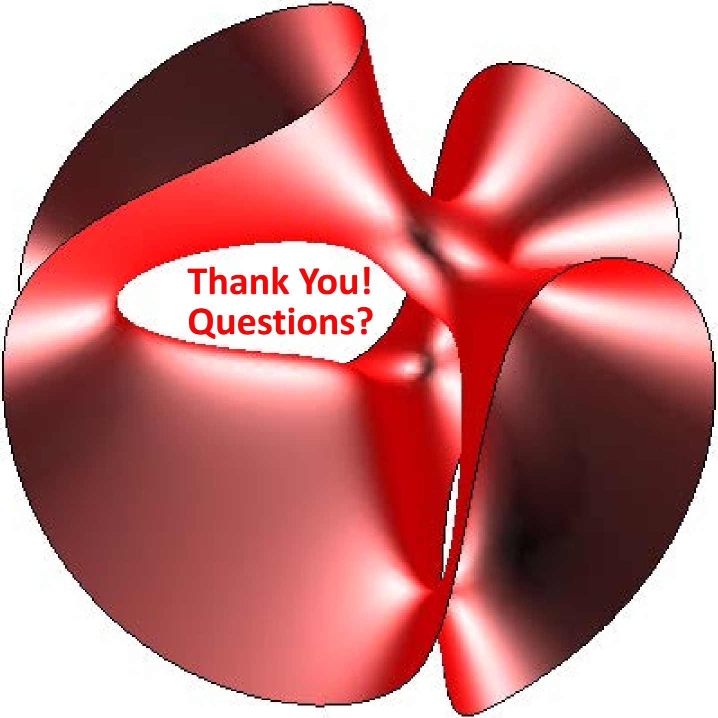
- The curve $y^p - y = mx^{p^h}$ defined over $\mathbb{F}_{p^{2h}}$, with $m^{p^h - 1} = -1$, is maximal over $\mathbb{F}_q = \mathbb{F}_{p^{2h}}$ (an example of unusually small genus).


Thank You!
Questions?