Modular Forms for Abelian Varieties

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Theorem (Modularity Theorem - Wiles 1994, BCDT 2000)

For an elliptic curve $E/\mathbb{Q}$ of conductor $N$, there exists a cusp form $f$ of weight 2 and level $N$ such that

$$L(E, s) = L(f, s).$$

(For a complete story of the modularity theorem and Fermat’s last theorem, see for example “Modular Forms and Fermats Last Theorem” by Cornell, Silverman, and Stevens.)
In this theorem, $f$ is a cusp form for the Hecke’s subgroup

$$\Gamma_0(N) = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \text{SL}_2(\mathbb{Z}) \mid c \equiv 0 \pmod{N} \right\}.$$

Hence,

$$f : \Gamma_0(N) \backslash \mathbb{H} \to \mathbb{C}.$$

$N$ is called the **level** of such a modular form.

Also, $f$ can be viewed as a function on a double coset space

$$f : \Gamma_0(N) \backslash \text{SL}_2(\mathbb{R}) / \text{SO}_2(\mathbb{R}) \to \mathbb{C}$$

where, $\text{SL}_2$ is the special linear group and $\text{SO}_2$ is the special orthogonal group.
Using **strong approximation theorem** one may convert $f$ to an automorphic representation $\phi_f$ of $GL_2(\mathbb{A}_\mathbb{Q})$:

$$
\phi_f : Z_{\mathbb{A}_\mathbb{Q}} GL_2(\mathbb{Q}) \backslash GL_2(\mathbb{A}_\mathbb{Q}) / K_{\infty} \times \prod_p K_p^N \rightarrow \mathbb{C},
$$

with trivial central character, where

- $Z_{\mathbb{A}_\mathbb{Q}}$ is the center of $GL_2(\mathbb{A}_\mathbb{Q})$
- $K_{\infty} = GO_2(\mathbb{R})$
- $K_p^N = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in GL_2(\mathbb{Z}_p) \mid c \equiv 0 \pmod{N} \right\}$.  


For abelian varieties, a generalization of such a modularity theorem is attributed to the Langlands program:

For an abelian variety $A/\mathbb{Q}$ of dimension $n$ there exists an automorphic representation $\phi$ on $\text{GSpin}_{2n+1}$ such that

$$L(A, s) = L(\phi, s).$$
Gross has a refinement of this conjecture for a special case:

**Conjecture (Gross, 2015)**

Let $A/\mathbb{Q}$ be an abelian variety of dimension $n$ with $\text{End}_\mathbb{Q}(A) = \mathbb{Z}$. Then there exists an automorphic representation on the split $\text{GSpin}_{2n+1}(\mathbb{A}_\mathbb{Q})$,

$$\phi : \text{GSpin}_{2n+1}(\mathbb{Q}) \backslash \text{GSpin}_{2n+1}(\mathbb{A}_\mathbb{Q}) / K_\infty \times \prod_p K_p^N \to \mathbb{C},$$

with explicit weight and level, such that

$$L(A, s) = L(\phi, s).$$

In that article, Gross works with $\text{SO}_{2n+1}$ instead of $G\text{Spin}_{2n+1}$.

In fact, by “renormalizing”, one may assume that $\phi$ lives on $\text{SO}_{2n+1}$:

**Conjecture (Gross, 2015)**

Let $A/\mathbb{Q}$ be an abelian variety of dimension $n$ with $\text{End}_\mathbb{Q}(A) = \mathbb{Z}$. Then there exists an automorphic representation on the split $\text{SO}_{2n+1}(\mathbb{A}_\mathbb{Q})$,

$$\phi : \text{SO}_{2n+1}(\mathbb{Q})\backslash \text{SO}_{2n+1}(\mathbb{A}_\mathbb{Q})/K_\infty \times \prod_p K_0(p^m) \to \mathbb{C},$$

with explicit weight and level, such that

$$L(A, s) = L(\phi, s).$$
Gross’ conjecture addresses the special orthogonal group $SO(\Lambda)$ for the $\mathbb{Z}$-lattice

$$\Lambda = \langle a_1, \ldots, a_n, c, b_n, \ldots, b_1 \rangle$$

with bilinear form $(−, −)$ with

$$(a_i, b_i) = 1, \quad (c, c) = 2,$$

and all other inner products equal to zero.

The corresponding $\textbf{GSpin}$ group is the group satisfying the short exact sequence

$$1 \to \text{GL}_1 \to \text{GSpin}(\Lambda) \to \text{SO}(\Lambda) \to 1.$$ 

It is a group of type $B_n$ whose “derived” subgroup $\text{Spin}(\Lambda)$ is the double cover of $\text{SO}(\Lambda)$.
Let $N \in \mathbb{N}$. Let $\Lambda(N)$ be the sublattice spanned by the vectors

$$\{a_1, \ldots, a_n, Nc, Nb_n, \ldots, Nb_1\}$$

over $\mathbb{Z}$. The appropriate bilinear form on this sublattice is $(-, -)/N$.

Gross determines the “level” of the automorphic representation $\phi$ as a group scheme $K_0(N)$:

**Proposition (Gross, 2015)**

There exists a group scheme $K_0(N)/\mathbb{Z}$ with generic fiber isomorphic to $SO_{2n+1}/\mathbb{Q}$ and special fiber at $p$ isomorphic to $SO_{2n+1}/\mathbb{F}_p$ if $p \nmid N$ and $SO_{2n}/\mathbb{F}_p$ if $p \mid N$. 

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When $n = 1$, $SO_3 \cong SL_2$ and $K_0(N)$ is conjugate to

$$\Gamma_0(N) = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in SL_2(\mathbb{Z}) \mid c \equiv 0 \pmod{N} \right\}.$$ 

When $n = 2$, $SO_5 \cong PGSp_4$ and $K_0(N)$ is conjugate to the “paramodular subgroups” defined by Roberts and Schmidt (see “Local newforms for $GSp_4$” by Roberts and Schmidt).
In my thesis, I have used Gross’ construction to determine the corresponding level for $G\text{Spin}(\Lambda)$:

**Proposition (S.)**

There exists a group scheme $G = G(\Lambda(N))/\mathbb{Z}$ with generic fiber $G_{\mathbb{Q}} \cong G\text{Spin}_{2n+1}/\mathbb{Q}$ and special fiber $G_{F_p} \cong G\text{Spin}_{2n+1}/F_p$ if $p \nmid N$ and $G_{F_p} \cong G\text{Spin}_{2n}/F_p$ if $p \mid N$ that satisfies the short exact sequence

$$1 \to GL_1/\mathbb{Z} \to G/\mathbb{Z} \to K_0(N)/\mathbb{Z} \to 1.$$
Gross gives an explicit recipe for constructing a global cusp form $F$, made from a tensor product of local cusp forms:

$$F : G(\mathbb{Q}) \backslash G\text{Spin}_{2n+1}(\mathbb{A}_{\mathbb{Q}})/K_\infty \times \prod_p G(\mathbb{Z}_p) \to \mathbb{C}.$$  

By strong approximation, $F$ is completely determined by restricting to its archimedean component

$$F_\infty : G(\mathbb{Q}) \backslash G\text{Spin}_{2n+1}(\mathbb{R})/K_\infty \to \mathbb{C}.$$
Determine $K_{\infty}$: Gross has also addressed the weight $K_{\infty}$ of the automorphic representation $\phi$ for $SO_{2n+1}$ explicitly. What is the corresponding weight for $GSpin_{2n+1}$?

Gross’ work is only focused on the case of trivial central character for the automorphic representation $\phi$. What if we had a nontrivial central character? In other words, I am interested in finding a class of $GSpin$ automorphic representations with an arbitrary central character whose restriction to the trivial central character is the work of Gross.
Cunningham and Dembélé have recently used lifts of Hilbert modular forms to general odd spin groups to construct nontrivial examples of abelian varieties that satisfy Gross’ conjecture:

\[
\begin{align*}
    f & \overset{\text{Arthur-Clozel}}{\rightarrow} \pi' \quad \text{on} \quad GL_{2n} \\
    & \overset{\text{Shahidi-et al.}}{\rightarrow} \pi' \quad \text{descends to} \quad GSpin_{2n+1}.
\end{align*}
\]
Thank You!