Constant symplectic 2-groupoids

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The integration problem for Courant algebroids

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This is joint work with Xiang Tang.

Differential forms on simplicial manifolds

Let $X_\bullet$ be a simplicial manifold. Let $f_i^q : X_q \to X_{q-1}$ denote the face maps, and let $\sigma_i^q : X_q \to X_{q+1}$ denote the degeneracy maps.

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$$\delta\alpha := \sum_{i=0}^{q+1} (-1)^i (f_i^{q+1})^* \alpha$$

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Definition

A differential form $\alpha \in \Omega^\bullet(X_q)$ is called

- *multiplicative* if $\delta\alpha = 0$,
- *normalized* if $(\sigma_{q-1}^i)^*\alpha = 0$ for all i.

The tangent complex

For $x \in X_0$ and $q \geq 0$, let $\sigma^q := \sigma_0^{q-1} \cdots \sigma_0^0$, and

$$T_{x,q}X := T_{\sigma^q(x)}X_q.$$

Then $T_{x,\bullet}X$ is a simplicial vector space.

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The *tangent complex* of $X_\bullet$ at x is

$$\cdots \to \hat{T}_{x,q}X \xrightarrow{\partial} \hat{T}_{x,q-1} \xrightarrow{\partial} \cdots \hat{T}_{x,0}X = T_x X_0.$$

Lie 2-groupoids

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The tangent complex of a Lie 2-groupoid vanishes above degree 2, so we have a 3-term complex of vector bundles

$$\hat{T}_2 X \xrightarrow{\partial} \hat{T}_1 X \xrightarrow{\partial} \hat{T}_0 X = TX_0.$$

Simplicial nondegeneracy

Let $X_\bullet$ be a Lie 2-groupoid, and let ω be a normalized 2-form on X_2. Define two associated pairings:

1. For $v \in T_x X_0$ and $w \in T_{x,2}X$,

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Definition

ω is *simplicially nondegenerate* if A_ω and B_ω are nondegenerate pairings for all $x \in X_0$.

Symplectic 2-groupoids

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If $\alpha \in \Omega^2(X_1)$ is closed, normalized, and satisfies $A_{\delta\alpha} = B_{\delta\alpha} = 0$, then $\omega' = \omega + \delta\alpha$ is considered equivalent to ω.

Linear 2-groupoids

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$$(W_2 \xrightarrow{\partial} W_1 \xrightarrow{\partial} W_0) \quad \leftrightarrow \quad \begin{array}{c} V_2 = W_2 \oplus W_1 \oplus W_1 \oplus W_0 \\ \downarrow\downarrow\downarrow \\ V_1 = W_1 \oplus W_0 \\ \downarrow\downarrow \\ V_0 = W_0 \end{array}$$

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So structures on linear 2-groupoids can be translated into structures on 3-term chain complexes.

Constant 2-forms

Theorem

There is a one-to-one correspondence between constant normalized multiplicative 2-forms $\omega \in \Omega(V_2)$ *and pairs* (C_{41}, C_{32})*, where* C_{41} *is a bilinear pairing of* W_0 *with* W_2 *and* C_{32} *is a bilinear form on* W_1 *such that*

$$C_{41}(\partial w_1, w_2) = C_{32}(\partial w_2, w_1) + C_{32}(w_1, \partial w_2).$$

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$$\omega = \begin{bmatrix} C_{11} & C_{12} & C_{13} & C_{14} \\ C_{21} & C_{22} & C_{23} & 0 \\ C_{31} & C_{32} & 0 & 0 \\ C_{41} & 0 & 0 & 0 \end{bmatrix}.$$

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Degeneracy vs simplicial nondegeneracy.

Minimal description of constant symplectic 2-groupoids

Theorem

There is a one-to-one correspondence between constant symplectic 2-groupoids and tuples $(W_1, W_0, \langle\cdot,\cdot\rangle, \partial, r)$*, where*

- W_1 *and* W_0 *are vector spaces,*
- $\langle\cdot,\cdot\rangle$ *is a nondegenerate symmetric bilinear form on* W_1*,*
- $\partial : W_1 \to W_0$ *is a linear map such that the image of* ∂^* *in* $W_1^* \cong W_1$ *is isotropic,*
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Furthermore, equivalences change r *arbitrarily and nothing else.*

When $r = 0$, we call the symplectic 2-groupoid *symmetric*. Note that in this case ω is genuinely nondegenerate.

Constant Courant algebroids

Given a (symmetric) constant symplectic 2-groupoid with data $(W_1, W_0, \langle\cdot,\cdot\rangle, \partial)$, we can form a Courant algebroid structure on $W_1 \times W_0 \to W_0$, where

- The bilinar form is $\langle\cdot,\cdot\rangle$,
- The anchor map $\rho : W_1 \times W_0 \to TW_0 = W_0 \times W_0$ is given by $\rho(w_1, w_0) = (\partial w_1, w_0)$,
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Theorem

There is a one-to-one correspondence between constant Courant algebroids and equivalence classes of constant symplectic 2-groupoids.

Linear Lagrangian sub-2-groupoids

Let $(V_\bullet, \omega)$ be a symmetric constant symplectic 2-groupoid with data $(W_1, W_0, \langle \cdot, \cdot \rangle, \partial)$.

Proposition

Linear Lagrangian sub-2-groupoids $L_\bullet \subseteq V_\bullet$ *are in one-to-one correspondence with pairs* (U_1, U_0), $U_i \subseteq W_i$, *such that* $U_1^\perp = U_1$ *and* $\partial U_1 \subseteq U_0$.

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In the case where $L_\bullet$ is *wide*, i.e. $U_0 = W_0$, then $U_1 \times W_0 \subseteq W_1 \times W_0$ is a Dirac structure. We call this a *constant Dirac structure*.

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Theorem

There is a one-to-one correspondence between constant Dirac structures and wide linear Lagrangian sub-2-groupoids.

Thanks!