Lie theory of multiplicative structures.

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Workshop on Geometric Structures on Lie groupoids - Banff, 2017

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- Unifying framework: general theory of multiplicative tensor fields (joint with H. Bursztyn).
- van Est for differential forms with coefficients (joint with A. Cabrera).

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$$\delta([u, v]) = [\delta(u), v] + [u, \delta(v)], \ \ u, v \in \mathfrak{g} \text{ (cocycle equation)}$$

$$\delta^2 = 0$$

⋄ Correspondence: $\mathcal{L}_{\vec{u}}\pi = \overrightarrow{\delta(u)}$

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Integration: Cocycles.

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Integration: Lie bialgebroid $\Leftrightarrow \pi_A : T^*A \to TA$ is a Lie algebroid morphism.

Examples

$\diamond$ Poisson manifolds. (M, π_M) Poisson manifold $\Rightarrow (T^*M, TM)$ is a Lie bialgebroid:

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is the de Rham differential. Integration yields the symplectic groupoid.

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In this case, $\delta = [r, \cdot]$.

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$\diamond$ Correspondence: $\quad i_{\vec{a}}\omega = t^*\mu(a)$

◇ (twisted) Symplectic:

(Cattaneo-Xu, 2004)

ω non-degenerate $\Leftrightarrow$ $\pi = \rho \circ \mu^{-1} : T^*M \to TM$ (twisted) Poisson structure

◇ Pre-symplectic:

(Bursztyn-Crainic-Weinstein-Zhu, 2004)

$$(*) \begin{cases} \dim(\mathcal{G}) = 2\dim(M) \\ \ker(\omega) \cap \ker(ds) \cap \ker(dt) = 0 \end{cases} \Leftrightarrow$$

$(\rho, \mu) : A \hookrightarrow TM \oplus T^*M$ (twisted) Dirac structure

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Recently, $(*)$ interpreted as $\omega : T\mathcal{G} \to T^*\mathcal{G}$ being a Morita map (Del Hoyo, Ortiz(2017)).

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Integration: Explicit construction of 2-form on the Weinstein groupoid $\mathcal{G}(A)$. (Cattaneo-Felder (2000) ; Crainic, Fernandes (2003))

Further works

◇ Kosmann-Schwarzbach: Multiplicativity, from Lie groups to generalized geometry. *preprint arXiv:1511.02491* (2015).

◇ Mackenzie, Xu: Classical lifting processes and multiplicative vector fields. *Quarterly J. Math.* (1998).

◇ Iglesias, Marrero: Jacobi groupoids and generalized Lie bialgebroids, *J. Geom. Phys* (2003).

◇Crainic, Zhu: Integrability of Jacobi and Poisson structures, *Ann. Inst. Fourier (Grenoble)* (2007).

◇ Grabowski, Rotkiewicz: Higher vector bundles and multi-graded symplectic manifolds, *J. Geom. Phys.* (2009).

◇ Laurent, Stienon, Xu: Integration of holomorphic Lie algebroids. *Math.Ann.* (2009).

◇ Arias Abad, Crainic: The Weil algebra and the Van Est isomorphism. *Ann. Inst. Fourier (Grenoble)* (2011).

◇ Bursztyn, Cabrera: Multiplicative forms at the infinitesimal level. *Math. Ann.* (2012).

◇ Iglesias, Laurent, Xu: Universal lifting and quasi-Poisson groupoids. *JEMS* (2012).

◇ Jotz Lean, Ortiz: Foliated groupoids and infinitesimal ideal systems, *Indag. Math.* (2014)

◇ Crainic, Salazar, Struchiner: Multiplicative forms and Spencer operators. *Math Z* (2015).

◇ Cabrera, Marcut, Salazar: A construction of local Lie groupoids using Lie algebroid sprays. *preprint arXiv:1703.04411* (2017).

General tensor fields

Idea: Treat tensor fields as functions on Whitney sum of tangent and cotangent bundles.

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(Linear data) $\mu : A \to \mathbb{R}, \quad \mu([a, b]) = \mathcal{L}_{\rho(a)}\mu(b) - \mathcal{L}_{\rho(b)}\mu(a)$.

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- Adaptable method to other contexts (e.g. van Est for differential forms with coefficients)

Cost: Work on different groupoid.

Compatibility

$\mathcal{G} \rightrightarrows M$ Lie groupoid and $\tau \in \Gamma(\wedge^p T^*\mathcal{G} \otimes \wedge^q T\mathcal{G})$ a (q, p) tensor field.

View τ as a *function* on $\mathbb{G} = (\oplus^p T\mathcal{G}) \oplus (\oplus^q T^*\mathcal{G})$,

$$(U_1, \ldots, U_p, \xi_1, \ldots, \xi_q) \xrightarrow{c_\tau} \tau(U_1, \ldots, U_p, \xi_1, \ldots, \xi_q).$$

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Definition: τ is *multiplicative* if $c_\tau \in C^\infty(\mathbb{G})$ is a multiplicative function.

Previous appearences in the literature:

◇ $(q, 0)$ case: multivector fields - (Iglesias, Laurent, Xu, 2012).

$$c_\pi \text{ multiplicative } \Leftrightarrow \left(\pi \oplus \pi \oplus (-1)^{q+1}\pi\right)(\xi_1, \ldots, \xi_q) = 0,$$

for $\xi_i \in N^*(\mathrm{graph}(m))$.

◇ $(0, p)$ case: differential forms - (Bursztyn, Cabrera, 2012)

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For vector-valued forms $\tau \in \Gamma(\wedge^q T^*\mathcal{G} \otimes T\mathcal{G})$, we have that τ is multiplicative $\Leftrightarrow$

$$\overline{\tau} : \underbrace{T\mathcal{G} \oplus \cdots \oplus T\mathcal{G}}_{q-times} \to T\mathcal{G} \textit{ is a groupoid morphism}$$

Infinitesimal components

Let $\mathbb{A}$ be the Lie algebroid of $\mathbb{G}$

Cocycle:

Multiplicative $\tau \in \Gamma(\wedge^p T^*\mathcal{G} \otimes \wedge^q T\mathcal{G}) \leftrightharpoons \mu \in \Gamma(\mathbb{A}^*), d\mu = 0$

$$\mathcal{L}_{\overrightarrow{\mathbb{a}}} c_\tau = \mathbb{t}^* \underbrace{\langle \mu, \mathbb{a} \rangle}_{\in C^\infty(\mathbb{M})}, \quad \text{for } \mathbb{a} \in \Gamma(\mathbb{A}).$$

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$\diamond$ $\Gamma(\mathbb{A})$ is generated by 3 types of sections $\mathbb{u}_a$, $\mathbb{v}_a$, $\mathbb{v}_\alpha$, parametrized by $a \in \Gamma(A)$, $\alpha \in \Omega^1(M)$ (DVB theory).

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Evaluation maps:

$$D : \Gamma(A) \to C^\infty(\mathbb{M}), \;\; D(a) = \langle \mu, \mathbb{u}_a \rangle$$

$$l : A \to C^\infty(\mathbb{M}), \;\; l(a) = \langle \mu, \mathbb{v}_a \rangle$$

$$r : T^*M \to C^\infty(\mathbb{M}), \;\; r(\alpha) = \langle \mu, \mathbb{v}_\alpha \rangle.$$

◇ $\underbrace{(D, l, r)}_{(0,0),(1,0),(0,1)}$ take values in $\Gamma(\wedge^{p-i} T^*M \otimes \wedge^{q-j} A) \subset C^\infty(\mathbb{M})$.

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◇ Pull-back by target map $\mathbb{t}^* : C^\infty(\mathbb{M}) \to C^\infty(\mathbb{G})$ restricts to

$$\begin{array}{rccc} \mathcal{T} : & \Gamma(\wedge^p T^* M \otimes \wedge^q A) & \to & \Gamma(\wedge^p T^* \mathcal{G} \otimes \wedge^q T\mathcal{G}) \\ & \beta \otimes \mathfrak{X} & \mapsto & t^* \beta \otimes \overrightarrow{\mathfrak{X}}. \end{array}$$

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What about $\mathcal{L}_{\overrightarrow{\mathfrak{a}}} c_\tau$?

$\diamond \quad \underbrace{(D, l, r)}_{(0,0),(1,0),(0,1)} \quad$ take values in $\Gamma(\wedge^{p-i} T^* M \otimes \wedge^{q-j} A) \subset C^\infty(\mathbb{M})$.

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So,

$$\mathcal{L}_{\vec{a}} \tau = \mathcal{T}(D(a)), \quad i_{\vec{a}} \tau = \mathcal{T}(l(a)), \quad i_{t^* \alpha} \tau = \mathcal{T}(r(\alpha)).$$

IM equations

$d\mu = 0 \Leftrightarrow (D, r, l)$ satisfy

(1) $D([a, b]) = a \cdot D(b) - b \cdot D(a)$

(2) $l([a, b]) = a \cdot l(b) - i_{\rho(b)} D(a)$

(3) $r(\mathcal{L}_{\rho(a)} \alpha) = a \cdot r(\alpha) - i_{\rho^*(\alpha)} D(a)$

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Here $\Gamma(A)$ acts on $\Gamma(\wedge^\bullet T^*M \otimes \wedge^\bullet A)$ via

$$a \cdot (\alpha \otimes b) = \mathcal{L}_{\rho(a)}\alpha \otimes b + \alpha \otimes [a, b]$$

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Theorem: (Bursztyn, D.)
There is 1-1 correspondence between
$\tau \in \Gamma(\wedge^q T\mathcal{G} \otimes \wedge^p T^*\mathcal{G})$ *multiplicative* and (D, l, r), where

$$D : \Gamma(A) \to \Gamma(\wedge^p T^*M \otimes \wedge^q A), \quad \text{Leibniz-like condition},$$

$$l : A \to \wedge^{p-1} T^*M \otimes \wedge^q A,$$

$$r : T^*M \to \wedge^p T^*M \otimes \wedge^{q-1} A,$$

satisfying (1)–(6).

$\diamond$ Multivector fields: $\Pi \in \Gamma(\wedge^q T\mathcal{G}) \leftrightharpoons \begin{cases} D : \Gamma(A) \to \Gamma(\wedge^q A) \\ r : T^*M \to \wedge^{q-1} A \end{cases}$

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Define $\delta : \Gamma(\wedge^\bullet A) \to \Gamma(\wedge^{\bullet+q-1} A)$ via

$$\delta_0(f) = (-1)^q r(df),\ \delta_1(a) = D(a)$$

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IM-equations equivalent to

$$\delta([a, b]) = [\delta(a), b] + (-1)^{(|a|-1)(q-1)}[a, \delta(b)].$$

So, infinitesimal components are q-derivations of the Gerstenhaber algebra $(\Lambda^\bullet A, [\cdot, \cdot], \wedge)$.

Iglesias, Laurent-Gengoux, Xu(2012)

◇ Differential forms: $\omega \in \Gamma(\wedge^p T\mathcal{G}) \leftrightharpoons \begin{cases} D : \Gamma(A) \to \Gamma(\wedge^p T^*M) \\ l : A \to \wedge^{p-1} T^*M \end{cases}$

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$$\begin{aligned} \nu([a,b]) &= \mathcal{L}_{\rho(a)}\nu(b) - i_{\rho(b)}d\nu(a) \\ \mu([a,b]) &= \mathcal{L}_{\rho(a)}\mu(b) - i_{\rho(b)}d\mu(a) - i_{\rho(b)}\nu(a) \\ \langle \mu(b), \rho(a) \rangle &= -\langle \mu(a), \rho(b) \rangle \end{aligned}$$

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Arias Abad-Crainic(2011) ; Bursztyn-Cabrera(2012) .

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Also,

$$d\omega = \mathrm{s}^*\phi - \mathrm{t}^*\phi \Leftrightarrow \nu(a) = i_{\rho(a)}\phi$$

◇ Vector valued forms

$$K \in \Gamma(\wedge^p T^*\mathcal{G} \otimes T\mathcal{G}) \leftrightharpoons \begin{cases} D : \Gamma(A) \to \Gamma(\wedge^p T^*M \otimes A) \\ l : A \to \wedge^{p-1} T^*M \otimes A \\ r : T^*M \to \wedge^p T^*M \end{cases}$$

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$\diamond$ Nijenhuis torsion: In this case $p = 1$ and $N_K = \frac{1}{2}[K, K]$ has infinitesimal components

$$D^2 : \Gamma(A) \to \Omega^2(M, A)$$

$$[D, l] : A \to T^*M \otimes A, \;\; [D, l](a)|_X = D_X(l(a)) - l_X(D(a))$$

$$N_r \in \Omega^2(M, TM).$$

$p = 1 \Rightarrow D : \Gamma(A) \to \Gamma(T^*M \otimes A),\;\; l : A \to A,\; r : TM \to TM.$

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$\diamond$ This framework can be applied to: Poisson quasi-Nijenhuis structures, multiplicative projections, almost product structures...

Coefficients

Tensors with values in a representation E are functions on

$$\underbrace{\mathbb{G} \oplus (E^* \rtimes \mathcal{G})}_{\mathbb{G}_E} \rightrightarrows \underbrace{\mathbb{M} \oplus E^*}_{\mathbb{M}_E}$$

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Infinitesimal-Global results

◇ Linear data: A cocycle μ on the Lie algebroid $\mathbb{A}_E$.

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◇ More general coefficients (Joint w/ L.Egea): Representation up to homotopy. Change $E^* \rtimes \mathcal{G}$ by VB-groupoids. (Ehresmann connection, multiplicative distributions, forms with values in the adjoint representation.)

van Est

Multiplicative functions are cocycles on a diff. complex $(C^\bullet(\mathcal{G}), \delta)$, where

$$C^k(\mathcal{G}) = C^\infty(\mathcal{G}^{(k)}), \;\text{ space of } k \text{ composable arrows.}$$
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The infinitesimal-global extends to

$$VE : C^\bullet(\mathcal{G}) \to \Gamma(\wedge^\bullet A^*).$$

$$VE(f)(u_1, \ldots, u_k) = \begin{array}{c}\text{“formula envolving Lie derivatives of } f \\ \text{along the r.i. vect. fields } \overrightarrow{u_i}\text{”.}\end{array}$$

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(van Est; Weinstein-Xu; Crainic): $\mathcal{G}$ source k-connected, VE defines isomorphisms on cohomology up to degree k.

van Est for forms with coefficients

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◇ Weil algebra: $W^{k,p}(A, E)$ is the space of sequences $(c_0, c_1, \dots)$

$$c_i : \underbrace{\Gamma(A) \times \cdots \times \Gamma(A)}_{k-i} \to \Omega^{p-i}(M, S^i A^* \otimes E)$$

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◇ $k = 1$: Elements of $W^{1,p}(A, E)$ are (c_0, c_1),

$$c_0 : \Gamma(A) \to \Omega^p(M, E), c_1 : A \to \wedge^{p-1} T^*M \otimes E$$

◇ Embedding:

$W^{k,p}(A, E) \hookrightarrow \Gamma(\wedge^k \mathbb{A}_E^*)$ through evaluation maps

$$c_i(a_1, \ldots, a_{k-i} | b_1, \ldots, b_i) = \mu(\mathbb{u}_{a_1}, \ldots, \mathbb{u}_{a_{k-i}}, \mathbb{v}_{b_1}, \ldots, \mathbb{v}_{b_i})$$

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Theorem: (Cabrera, D.) If $\mathcal{G}$ is source p-connected, the van Est map VE for $\mathbb{G}_E$ induces isomorphism in cohomology

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for $k \leq p$.

◇ Explicit formulas:

$$\mathcal{L}_{\overrightarrow{\mathbb{u}\mathrm{a}}} \rightsquigarrow \mathcal{L}_{\overrightarrow{a}}, \quad \mathcal{L}_{\overrightarrow{\mathbb{v}\mathrm{a}}} \rightsquigarrow i_{\overrightarrow{a}}$$

Thank you.