Surface Instabilities in Nonlinear Elasticity

Joel Cawte

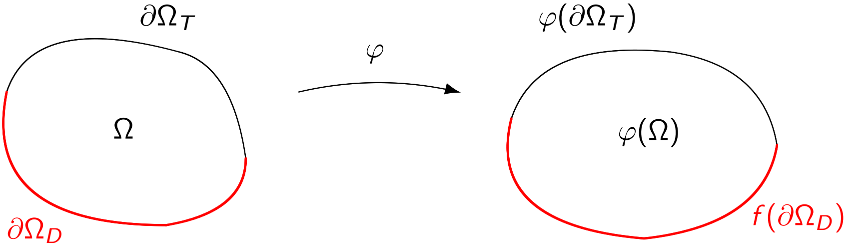
University of Bath

27th November, 2017

Notation

- Consider an elastic body occupying a domain $\Omega \subset \mathbb{R}^n$ in its reference configuration. Let $\varphi \in W^{1,2}(\Omega, \mathbb{R}^n)$ be a deformation of the body, with $\det(\nabla\varphi) > 0$, subject to the mixed displacement/traction condition

$$\varphi|_{\partial\Omega_D} = f,$$
$$\partial\Omega = \partial\Omega_D \cup \partial\Omega_T, \qquad \partial\Omega_D \cap \partial\Omega_T = \emptyset.$$



Hyperelasticity

- We assume the material is Hyperelastic, so we can associate an energy with each deformation φ given by

$$E[\varphi] = \int_\Omega W(x, \nabla\varphi(x))\,\mathrm{d}x,$$

where $W : \Omega \times \mathrm{M}_+^{n\times n} \to \mathbb{R}$ is the Stored Energy Function.

- We shall consider necessary conditions for $\varphi \in W^{1,2}(\Omega, \mathbb{R}^n)$ to be a strong or weak local minimiser.
- Incompressible Elasticity includes the restriction $\det(\nabla\varphi) = 1$, and $W : \Omega \times \mathrm{M}_1^{n\times n} \to \mathbb{R}$.

Weak local minimisers

- If φ is a sufficiently smooth solution to the Euler Lagrange equations, a further necessary condition for it to be a weak local minimiser is that the *second variation* at φ

$$\delta^2 E[\varphi](u) = \int_\Omega C[\nabla u, \nabla u] \, \mathrm{d}x$$

is nonnegative for all variations $u \in W^{1,2}_{\partial\Omega_D}(\Omega, \mathbb{R}^n)$, where $C^{ij}_{\alpha\beta} = \frac{\partial^2 W(x, \nabla\varphi(x))}{\partial F_{i\alpha} \partial F_{j\beta}}$.

The Complementing Condition

- Let $x_0 \in \partial\Omega_T$, and let ν be the unit normal at x_0. Write $H_\nu = \{x \in \mathbb{R}^n \mid x \cdot \nu < 0\}$, and $C_0 = \frac{\partial^2 W(x_0, \nabla\varphi(x_0))}{\partial F^2}$. Consider the boundary-value problem:

$$\begin{aligned} \operatorname{div}(C_0[\nabla u]) = 0 \quad & \text{in } H_\nu \\ C_0[\nabla u]\nu = 0 \quad & \text{on } \partial H_\nu. \end{aligned} \tag{1}$$

Definition

We say the boundary-value problem (1) satisfies the *complementing condition* if the only bounded solutions of the form

$$u = \mathrm{Re}(f(x\cdot\nu)\mathrm{e}^{i(x\cdot\tau)}), \qquad \tau \perp \nu \tag{2}$$

for (1) are trivial.

Agmon's Condition

- Consider the related boundary-value problem:

$$\begin{aligned} \operatorname{div}\left(C_0[\nabla u]\right) = \alpha^2 u \quad & \text{in } H_\nu \\ C_0[\nabla u]\nu = 0 \quad & \text{on } \partial H_\nu. \end{aligned} \tag{3}$$

Definition

We say the boundary-value problem (1) satisfies *Agmon's condition* if the only bounded solutions of the form (2) for (3) with $\alpha \neq 0$ are trivial.

Definition

The boundary-value problem (1) satisfies the *strong complementing condition* if it satisfies the complementing condition and Agmon's condition.

Quasiconvexity at the Boundary

Definition

For a free boundary point $x_0 \in \partial\Omega_T$ with normal ν, a *standard boundary domain* is a bounded domain $D_\nu \subset H_\nu$, such that the interior Γ of $\partial D_\nu \cap \partial H_\nu$ is non-empty.

Definition

The stored energy function W is *quasiconvex at the boundary at* φ (see Ball and Marsden [1984]) if for all free boundary points $x_0 \in \partial\Omega_T$ with normal ν, and any standard boundary domain $D_\nu \subset \mathbb{R}^n$,

$$\int_{D_\nu} W(x_0, \nabla\varphi(x_0) + \nabla\psi(x))\,\mathrm{d}x \geq \int_{D_\nu} W(x_0, \nabla\varphi(x_0))\,\mathrm{d}x,$$

for all $\psi \in W^{1,\infty}_{\partial D_\nu \setminus \Gamma}(D_\nu, \mathbb{R}^n)$

Biot Instability

- Biot [1963] looked for instabilities when $n = 2$, in an incompressible, Neo-Hookean material occupying H_ν, with $\nu = e_2$, by seeking solutions to the linearized equations around the homogeneous deformation

$$\varphi = \begin{pmatrix} \lambda_1 x_1 \\ \lambda_2 x_2 \end{pmatrix}$$

$$\lambda_1 \lambda_2 = 1.$$

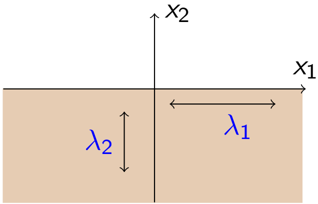
- Predicts surface instabilities at a compression ratio of $\frac{\lambda_1}{\lambda_2} \approx 0.544$.

Biot Instability

- Biot [1963] looked for instabilities when $n = 2$, in an incompressible, Neo-Hookean material occupying H_ν, with $\nu = e_2$, by seeking solutions to the linearized equations around the homogeneous deformation

$$\varphi = \begin{pmatrix} \lambda_1 x_1 \\ \lambda_2 x_2 \end{pmatrix}$$

$$\lambda_1 \lambda_2 = 1.$$



- Predicts surface instabilities at a compression ratio of $\frac{\lambda_1}{\lambda_2} \approx 0.544$.

Generalisation to Isotropic Materials

- Biot's original result follows if one were to formally check for failure of the complementing condition for a *Neo-Hookean, incompressible* stored-energy function:

$$W^{inc}(\nabla\varphi) = \underbrace{\frac{\mu}{2}|\nabla\varphi|^2}_{\text{Neo-Hookean part}} - \underbrace{p(x)\det\big((\nabla\varphi) - 1\big)}_{\text{Lagrange multiplier}}.$$

with the incompressibility condition $\det(\nabla\varphi) = 1$.

Theorem

Ball [1984] *Let* $W : D \to \mathbb{R}$ *be isotropic, and let* $\Phi : (0,\infty)^n \to \mathbb{R}$ *be the symmetric function given by* $W(F) = \Phi(v_1, \ldots, v_n)\ \ \forall F \in \mathrm{M}_+^{n\times n}$*, where* $v_1, \ldots, v_n$ *are the principal stretches of* F*. Then if* $F = \mathrm{diag}(v_1, \ldots, v_n)$*,* $G \in M^{n\times n}$*, and* $\Phi \in C^2((0,\infty)^n)$*, then*

$$\frac{\partial^2 W(F)}{\partial F^2}[G,G] = \sum_{i,j=1}^{n} \Phi_{,ij}(v) G_{ii} G_{jj} + \sum_{i\neq j} \frac{v_i \Phi_{,i}(v) - v_j \Phi_{,j}(v)}{v_i^2 - v_j^2} G_{ij}^2 + \frac{v_j \Phi_{,i}(v) - v_i \Phi_{,j}(v)}{v_i^2 - v_j^2} G_{ij} G_{ji}.$$

Generalisation to Isotropic Materials

- With the aid of this result, for a general isotropic, incompressible stored-energy function, instability occurs when

$$\alpha(r^3 - 2r^2 - r) - 2\beta r - \Phi_{22} r^2 + 2\Phi_{12} r - \Phi_{11} = 0,$$

where $r = \frac{\lambda_2}{\lambda_1}$, $\alpha = \frac{\lambda_2 \Phi_2 - \lambda_1 \Phi_1}{\lambda_2^2 - \lambda_1^2}$, and $\beta = \frac{\lambda_2 \Phi_1 - \lambda_1 \Phi_2}{\lambda_2^2 - \lambda_1^2}$.

Generalisation to Isotropic Materials

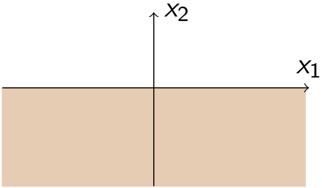
- We can compare this condition to the following: The homogeneous deformation $\varphi = (\lambda_1 x_1, \lambda_2 x_2)^T$ is a weak local minimiser only if

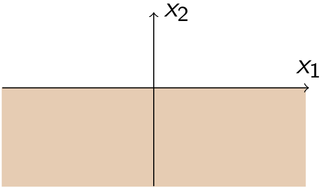
$$\alpha(\Phi_{11}\Phi_{22} - \Phi_{12}^2) + (\alpha^2 - \beta^2)\sqrt{\Phi_{11}\Phi_{22}} \geq 0$$

- Obtained by using Riccati Equations and a clever use of null lagrangians, applied to an isotropic, *compressible* material (see Mielke and Sprenger [1998]).

Gent and Cho [1999]

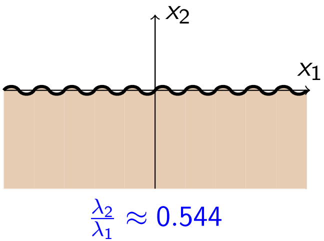
- Instabilities in the form of surface *creasing* have been observed to occur at a ratio of approximately 0.65, *before* wrinkling could occur. See Gent and Cho [1999].

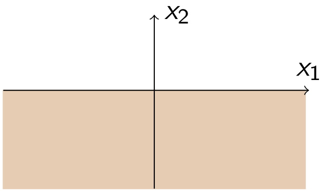




Gent and Cho [1999]

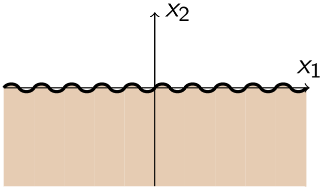
- Instabilities in the form of surface *creasing* have been observed to occur at a ratio of approximately 0.65, *before* wrinkling could occur. See Gent and Cho [1999].



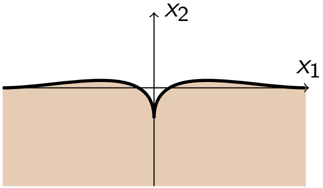


Gent and Cho [1999]

- Instabilities in the form of surface *creasing* have been observed to occur at a ratio of approximately 0.65, *before* wrinkling could occur. See Gent and Cho [1999].



$\frac{\lambda_2}{\lambda_1} \approx 0.544$



$\frac{\lambda_2}{\lambda_1} \approx 0.65$

Crease Formation

- Case study: Creasing in rubber elastomers under extreme circumstances

Figure: A sulcus on the interior of a rubber diaphragm

References

- J.M. Ball. Differentiabilty properties of symmetric and isotropic functions. *Duke Math. J.* **51** (1984) 699-728.
- J.M. Ball and J.E. Marsden. Quasiconvexity at the boundary, positivity of the second variation and elastic stability. *Arch. Rational Mech. Anal.* **86** (1984) 251-277
- M.A. Biot. Surface instability of rubber in compression. *Appli. Sci. Res.* **12** (1963) 168-182.
- A.N. Gent and I.S. Cho. Surface instabilities in compressed or bent rubber blocks. *Rubber Chem. Technol.* **72** (1999) 253-262.
- A. Mielke and P. Sprenger. Quasiconvexity at the boundary and a simple variational formulation of Agmon's condition. *Journal of Elasticity* **51** (1998) 23-41.