Calderón-Zygmund theory for nonlinear partial differential equations

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Outline

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IV. Calderón-Zygmund regularity results for equations with singular drifts.

I. Review on Calderón-Zygmund regularity theory and research problems

Review of regularity theory for PDE

- Calderón-Zygmund's theory (1952): If u is a weak solution of the linear equation

$$u_t - \mathrm{div}[\mathbb{A}_0(x,t)\nabla u)] = \mathrm{div}[F], \quad \text{in} \quad Q_2 := B_2 \times (-4,4),$$

and if $\mathbb{A}_0$ is uniformly elliptic and continuous, it holds that ($q \geq 2$)

$$\left(\fint_{Q_1} |\nabla u|^q dxdt\right)^{1/q} \leq C(n,q)\left\{\left(\fint_{Q_2} |\nabla u|^2 dxdt\right)^{1/2} + \left(\fint_{Q_2} |F|^q dxdt\right)^{1/q}\right\}$$

- Remark 1: The class of equations, and the estimates are invariant under the scalings and dilations:

$$u \mapsto \hat{u} := u/\lambda, \quad \text{and} \quad u \mapsto u_r(x,t) := u(rx, r^2 t)/r.$$

- Remark 2: Techniques in the proof rely on the scalings and dilations.

Review (cont.)

- The C-Z theory has been extended to the class of nonlinear equations
$$u_t - \mathrm{div}[\mathbb{A}(x,t,\nabla u)] = \mathrm{div}[F]$$
Refs: F. Chiarenza-M. Frasca-A. Longo (1991), L. Caffarreli-I. Peral (1998); Maugeri - D.K. Pagalachev - L. G. Softova (2003), E. Acerbi- G. Mingione (2007), N. Krylov (2000s), S. Byun - L. Wang (2004 -2017),...

- Though nonlinear, this class of equations is invariant under the scalings and dilations.

- This is essential because the proof relies on the scalings and dilations.

Goals

- Goals: Establish estimates of C-Z type for more general class of nonlinear (elliptic/parabolic) equations

$$u_t - \text{div}[\mathbb{A}(x, t, u, \nabla u) + \mathbf{b}u] = \text{div}[F] + f.$$

- Obvious issue: Due to the dependent of $\mathbb{A}$ on u, this class of equations is not invariant under the scalings and the dilations.

II. Calderón-Zygmund regularity results for cross-diffusion systems

A class of nonlinear diffusion equations

- Consider the equation (Gurney and Nisbet - 1975)

  $v_t - \mathrm{div}[(1+\gamma v)\nabla v] = v[1 - g(x,t) - v] \quad \text{in} \quad \Omega_T := \Omega \times (0,T),$

  where $g : \Omega_T \to \mathbb{R}$ is a given measurable function, and $\gamma \geq 0$.

- When $\gamma = 0$, the equation is just the standard reaction-diffusion equation.

- Remark 1: In our setting $\mathbb{A}(x,t,v,\nabla v) = (1+\gamma v)\nabla v$. Hence, $\mathbb{A}$ depends on v as its variable.

- Remark 2: If $g \in L^q$, with $q > (n+2)/2$, the De Giorgi-Nash-Moser theory implies that bounded solution v is C^α.

- Goal: To control $\|\nabla v\|_{L^q}$ by $\|g\|_{L^q}$. This is important b/c when $q \leq (n+2)/2$ the De Giorgi- Nash-Moser theory does not apply.

Nonlinear C-Z estimate

Theorem (L. Hoang, T. Nguyen, T. P. - SIMA (2015))

Let $\mathbb{A}_0$ be uniformly elliptic measurable matrix, $v_0 \in L^\infty(\Omega)$ be non-negative, and let $g \in L^q(\Omega_T)$ be non-negative with $q \geq 2$. There exists unique weak bounded solution v of

$$v_t = \operatorname{div}[(1+v)\mathbb{A}_0(x,t)\nabla v] + v[1 - g(x,t) - v] \quad \text{in} \quad \Omega \times (0,T)$$

with homogeneous Neumann boundary condition and with $v(\cdot,0) = v_0(\cdot)$. Moreover, if $[[\mathbb{A}_0]]_{\mathrm{BMO}} \ll 1$, then for $\bar{t} \in (0,T)$

$$\|\nabla v\|_{L^q(\Omega\times(\bar{t},T))} \leq C(n, T, \bar{t}, q, \|v\|_{L^\infty}) \Big[1 + \|g\|_{L^q(\Omega\times(0,T))}\Big].$$

Proof.

Key ingredients: Perturbation technique, “scaling parameter equation technique”, and maximum principle. □

An application: SKT cross-diffusion model

- Let $u(x,t)$ and $v(x,t)$ be two physical/biological quantities (population densities).
- The two species compete, diffuse randomly and move to avoid overcrowding.
- *Shigesada-Kawasaki-Teramoto* model (1979):

$$\begin{cases} u_t = & \Delta[(d_1 + a_{11}u + a_{12}v)u] & +u(a_1 - b_1 u - c_1 v), & \Omega \times (0,T), \\ v_t = & \Delta[(d_2 + a_{22}v)v] & +v(a_2 - b_2 u - c_2 v), & \Omega \times (0,T), \end{cases}$$

with the initial condition

$$u(\cdot,0) = u_0(\cdot) \geq 0, \quad v(\cdot,0) = v_0(\cdot) \geq 0 \quad \text{in } \Omega \subset \mathbb{R}^n.$$

- $d_k, a_k, b_k, c_k > 0$ and $a_{ij} \geq 0$ are constants.

A brief history

- H. Amann (series of 3 papers (1988 - 1990)): For $q > n$, and initial data $u_0, v_0 \in W^{1,q}(\Omega)$, there is $T_{\max} > 0$ such that the system has unique non-negative solution:

$$(u, v) \in C((0, T_{\max}), W^{1,q}(\Omega) \times W^{1,q}(\Omega)).$$

- Lou-Ni-Wu (1998): Unique global-time smooth solution when $n = 2$
- Other work: Y. Choi-R. Lui-Y. Yamada (2003-2004), D. Le (2003-2005), P. During, A. Jungel, J. U. Kim, S.-A. Shim, A. Yagi, ...: Restrictive conditions on the coefficients or $n \leq 4$.
- T. P. (2007-2008): $n \leq 9$.
- When $n \geq 10$, the existence of global-time smooth solutions was an open question.

Global-time smooth solutions of the SKT system

Theorem (L. Hoang, T. Nguyen, T. P. - SIMA (2015))

Let $\Omega \subset \mathbb{R}^n$ be open, bounded and smooth for any $n \geq 2$, and let $u_0, v_0 \in W^{1,q}(\Omega)$ and non-negative with $q > n$. Then, the system

$$\begin{cases} u_t = & \Delta[(d_1 + a_{11}u + a_{12}v)u] & +u(a_1 - b_1 u - c_1 v), & \Omega \times (0,\infty), \\ v_t = & \Delta[(d_2 + a_{22}v)v] & +v(a_2 - b_2 u - c_2 v), & \Omega \times (0,\infty), \end{cases}$$

together with homogeneous Neumann boundary conditions and initial conditions $u(x,0) = u_0(x), v(x,0) = v_0(x)$ has unique, global-time smooth solution u, v with

$$u, v \in C([0,\infty); W^{1,q}(\Omega)) \cap C^\infty(\overline{\Omega} \times (0,\infty)).$$

Note: This theorem solves an open problem initiated by H. Amann (1988), and asked by Y. Yamada (2009).

Main issues

- Our system of equations:

$$\begin{cases} u_t = & \Delta[(d_1 + a_{11}u + a_{12}v)u] \quad +u(a_1 - b_1 u - c_1 v), \\ v_t = & \Delta[(d_2 + a_{22}v)v] \quad +v(a_2 - b_2 u - c_2 v). \end{cases}$$

- By H. Amann's theorem, we need to control the local-time solution (with $q > n$)

$$\sup_{0<t<T} \Big[\big\|u(\cdot,t)\big\|_{W^{1,q}(\Omega)} + \big\|v(\cdot,t)\big\|_{W^{1,q}(\Omega)} \Big] < \infty?$$

- When $n = 2$, Lou-Ni-Wu carefully used energy estimates, interpolation inequalities to obtain the estimate with some $q > 2$.

- The problem is more challenging for large n.

Our approach

- The first equation can be rewritten as

$$u_t = \mathrm{div}[(d_1 + 2a_{11}u + a_{12}v)\nabla u + a_{12}u\nabla v] + u(a_1 - b_1 u - c_1 v).$$

Lemma 1: There exists $\alpha = \alpha(n) > 1$ such that

$$\|u\|_{L^{\alpha q}(\Omega_T)} \leq C(n,q,T)\left[1 + \|\nabla v\|_{L^q(\Omega_T)}\right], \quad \forall q \geq 2.$$

- The second equation is rewritten as

$$v_t = \mathrm{div}[(d_2 + 2a_{22}v)\nabla v] + v(a_2 - b_2 u - c_2 v).$$

This is the Gurney-Nisbet equation that we considered.
Lemma 2: Our C-Z theorem gives

$$\|\nabla v\|_{L^q(\Omega_T)} \leq C(n,T,q)\left[1 + \|u\|_{L^q(\Omega_T)}\right], \quad \forall\ q \geq 2.$$

- Iterate the two lemmas, we can control $\|u\|_{L^q(\Omega_T)}, \|\nabla v\|_{L^q(\Omega_T)}$ for q that is as large as we want.

Summary (of Part II)

Summary:

- C-Z estimate is established for bounded, weak solutions of

$$v_t = \mathrm{div}[(1+v)\mathbb{A}_0(x,t)\nabla v] + v(1-g-v), \quad \text{in} \quad \Omega \times (0,T)$$

- The theory is applied to solve an open problem for the SKT system.

Questions:

- $W^{1,q}$-regularity theory for bounded weak solutions of more general class of equations

$$u_t = \mathrm{div}[\mathbb{A}(x,t,u,\nabla u)] + \mathrm{div}[F] + f(x,t).$$

(Note: Only C^α and $C^{1,\alpha}$ theory is extensively studied.)

- Can we relax the boundedness assumption on the solutions?

III. Calderón-Zygmund regularity results for general nonlinear p-Laplacian equations

Nonlinear elliptic equations

- Consider the equation

$$-\text{div}[\mathbb{A}(x, u, \nabla u)] = \text{div}[F] + f, \quad \text{in} \quad B_2.$$

- Assume that there is $p > 1$ and $\Lambda > 0$ such that

$$\Lambda^{-1}|\xi|^p \leq \langle \mathbb{A}(x, u, \xi)\xi, \xi \rangle, \quad |\mathbb{A}(x, u, \xi)| \leq \Lambda |\xi|^{p-1}.$$

  i.e. $\mathbb{A}$ grows in ξ as p-Laplacian.

- Motivation: calculus of variations, porous media, homogenization, geometric analysis, mathematical biology,...

- Goal: Develop the theory to estimate ∇u in L^q-spaces.

- Issues: Maximum principle (comparison principle), and invariance under scalings and dilations.

Theorem (T. Nguyen - T. P. - CVPDE (2016); T. P. (submitted))

Let $q \geq p > 1$ and $M_0 > 0$. Then, there exists $\delta = \delta(p, q, n) > 0$ such that if $\mathbb{A}$ grows as p-Laplacian, $[[\mathbb{A}]]_{\mathrm{BMO}} \leq \delta$, then for every weak solution of

$$-\mathrm{div}[\mathbb{A}(x, u, \nabla u)] = \mathrm{div}[F] + f, \quad in \quad B_2$$

satisfying $[[u]]_{\mathrm{BMO}(B_1)} \leq M_0$, it holds that

$$\|\nabla u\|_{L^q(B_1)}^{p-1} \leq C(p, q, n, M_0)\left[\|F\|_{L^{\frac{q}{p-1}}(B_2)} + \|f\|_{L^{\frac{qn}{q+n(p-1)}}(B_2)} + \|u\|_{L^p(B_2)}^{p-1}\right]$$

- Remark 1: Boundary regularity theory, global regularity theory for both elliptic and and parabolic p-Laplacian type equations are also obtained.
- Remark 2: The results recover known results in which $\mathbb{A}$ is independent on u.

Remarks

- In T. Nguyen - T. P. (CVPDE-2016), it is assumed that $\|u\|_{L^\infty(B_1)} \leq M_0$. We used comparison principle (J. Leray-J.-L. Lions (1965), J.-L. Lions(1969)).
- My new papers develop the theory to the borderline case: replacing $\|u\|_{L^\infty(B_1)} \leq M_0$ by $\|u\|_{\mathrm{BMO}} \leq M_0$.
  - A new approach, which does not rely on comparison principle, is introduced. Many regularity conditions on $\mathbb{A}$ are relaxed/dropped.
  - The theorem is important in critical cases. For example, in the study of n-Laplacian equations, the weak solutions are in $W^{1,n}$, so they are in BMO already.
- This regularity theory for BMO-solutions is completely new.

Technique: Equations with scaling parameter

- Instead of studying the class of equations

$$-\mathrm{div}[\mathbb{A}(x, u, \nabla u)] = \mathrm{div}[F] + f, \quad \text{in} \quad B_2.$$

- We study the family of equations with scaling parameter

$$-\mathrm{div}[\mathbb{A}(x, \lambda u, \nabla u)] = \mathrm{div}[F] + f, \quad \text{in} \quad B_2, \quad \lambda \geq 0.$$

- This class of equations is invariant under the scalings and dilations. For example, if u is a solution, then $\hat{u} = \frac{u}{\tau}$ is a solution of

$$-\mathrm{div}[\hat{\mathbb{A}}(x, \hat{\lambda}\hat{u}, \nabla \hat{u})] = \mathrm{div}[\hat{F}] + \hat{f}, \quad \text{in} \quad B_2,$$

where $\hat{\lambda} = \lambda\tau$, $\hat{\mathbb{A}}(x, u, \xi) = \mathbb{A}(x, u, \tau\xi)/\tau^{p-1}$, $\hat{F} = F/\tau^{p-1}$ and $\hat{f} = f/\tau^{p-1}$.
- Note 1: $\hat{\mathbb{A}}$ satisfies the same conditions as $\mathbb{A}$ (with the same ellipticity constant Λ).
- Note 2: $\|\lambda u\|_{L^\infty}$ is invariant under the scalings and dilations:

$$\left\|\hat{\lambda}\hat{u}\right\|_{L^\infty} = \|\lambda u\|_{L^\infty}.$$

IV. Calderón-Zygmund regularity results for equations with singular drifts

Equations with singular divergence-free drifts

- Let $\mathbb{A}_0$ be symmetric matrix. Consider the equation with drifts:

$$u_t - \text{div}[\mathbb{A}_0(x,t)\nabla u + \mathbf{b}u] = \text{div}[F] + f,$$

  where the drift vector field $\mathbf{b}$ is divergence-free.

- Refs: V. Liskevich-Y. Semenov (2000), H. Berestycki-F. Hamel-N. Nadirashvili (2005), L. A. Caffarelli - A. Vasseur (2010),...
- Under some conditions,

$$\mathbf{b}(x,t) = \text{div}[D(x,t)],$$

  where D is a skew-symmetric matrix, i.e. $D^* = -D$.

- In this case, the equation becomes

$$u_t - \text{div}[\mathbb{A}(x,t)\nabla u] = \text{div}[F] + f,$$

  where $\mathbb{A}(x,t) = \mathbb{A}_0(x,t) + D(x,t)$.

Existence - uniqueness result

Theorem (T. P. - JDE (2017))

Let $\mathbb{A} = \mathbb{A}_0 + D$ where $\mathbb{A}_0$ is symmetric and uniformly elliptic, and $D \in L_t^\infty(\mathrm{BMO})$. Then, for $u_0 \in L^2(\Omega_T)$ and $f \in L^2((0,T), H_0^{-1}(\Omega))$, there exists unique weak solution u of

$$\begin{cases} u_t &= \mathrm{div}[\mathbb{A}(x,t)\nabla u] + f, \quad & in \quad \Omega_T = \Omega \times (0,T), \\ u &= 0, & on \quad \partial\Omega \times (0,T), \\ u(\cdot,0) &= u_0(\cdot), & on \quad \Omega. \end{cases}$$

and the solution satisfies the usual energy estimate

$$\sup_{t\in(0,T)} \left\|u(\cdot,t)\right\|_{L^2(\Omega)} + \|\nabla u\|_{L^2(\Omega_T)} + \|u_t\|_{L^2((0,T),H_0^{-1}(\Omega))} \leq C\left[\|f\|_{L^2((0,T),H_0^{-1}(\Omega))} + \|u_0\|_{L^2(\Omega)}\right].$$

Note: $\mathbb{A}$ is singular b/c $\|D\|_{L^\infty(\Omega_T)}$ could be ∞. When $D = 0$, the theorem recovers the classical results.

Key ideas in the proof

- We need to show that the bilinear form $a : H_0^1(\Omega) \times H_0^1(\Omega)$ defined by
$$a(u,v) = \int_\Omega \langle \mathbb{A}(x,t)\nabla u, \nabla v\rangle dx$$
is coercive and bounded (uniformly in time).
- Note that $\mathbb{A} = \mathbb{A}_0 + D$. Since $D^* = -D$, we see that
$$\begin{aligned} a(u,u) &= \int_\Omega \langle [\mathbb{A}_0(x,t) + D(x,t)]\nabla u, \nabla u\rangle dx \\ &= \int_\Omega \langle \mathbb{A}_0(x,t)\nabla u, \nabla u\rangle dx \geq \Lambda^{-1} \|\nabla u\|_{L^2}^2 . \end{aligned}$$
- Also, by the boundedness of $\mathbb{A}_0$
$$\left| \int_\Omega \langle \mathbb{A}_0(x,t)\nabla u, \nabla v\rangle dx \right| \leq \Lambda \|\nabla u\|_{L^2} \|\nabla v\|_{L^2} .$$

Key ideas in the proof (cont.)

- It remains to prove that (this is the most delicate part)

$$\left| \int_\Omega \langle D(x,t)\nabla u, \nabla v \rangle dx \right| \leq C \, \|\nabla u\|_{L^2} \, \|\nabla v\|_{L^2} \; ?$$

- For simplicity, consider $n = 2$. Since $D^* = -D$, we can write

$$D = \begin{pmatrix} 0 & d(x,t) \\ -d(x,t) & 0 \end{pmatrix}, \quad \text{with} \quad d \in L_t^\infty(\mathrm{BMO})$$

- Then, $\langle D(x,t)\nabla u, \nabla v \rangle = d(x,t)[u_{x_1} v_{x_2} - u_{x_2} v_{x_1}]$, and note that

$$u_{x_1} v_{x_2} - u_{x_2} v_{x_1} = \det \begin{pmatrix} u_{x_1} & u_{x_2} \\ v_{x_1} & v_{x_2} \end{pmatrix}.$$

- By the div-curl theorem of R. Coifman, P.-L. Lions, Y. Meyer, S. Semmes (1993), $u_{x_1} v_{x_2} - u_{x_2} v_{x_1}$ is in the Hardy space $\mathcal{H}^1$.

- By C. Fefferman (1971) (see also C. Fefferman - E. Stein (1972)): The dual of $\mathcal{H}^1$ is BMO.

Regularity theory for equations with singular drifts

Theorem (T. P. - JDE (2017))

Let $\mathbb{A} = \mathbb{A}_0 + D$ *as before. For* $q \geq 2$*, there exists* $\delta = \delta(n, q) > 0$ *such that if* $[[\mathbb{A}]]_{\mathrm{BMO}} \leq \delta$ *and if* u *is a weak solution of*

$$\begin{cases} u_t & = & \mathrm{div}[\tilde{\mathbb{A}}(x, t, u, \nabla u)] + \mathrm{div}(F), & \text{in} \quad \Omega_T = \Omega \times (0, T), \\ u & = & 0, & \text{on} \quad \partial\Omega \times (0, T), \\ u(\cdot, 0) & = & u_0(\cdot), \quad \text{on} \quad \Omega. & \end{cases}$$

where $\tilde{\mathbb{A}}(x, t, u, \xi) \sim \mathbb{A}(x, t)\xi$ *when* $|\xi| \to \infty$ *(i.e. asymptotically Uhlenbeck), it holds that*

$$\|\nabla u\|_{L^q(\Omega \times (\bar{t}, T))} \leq C(n, \bar{t}, T, q)[1 + \|F\|_{L^q(\Omega_T)}].$$

- This theorem is new even for linear equations.
- It provides the $W^{1,q}$-analogue of C^α-theory (linear equations): Q.-S. Zhang (2004); G. Seregin - L. Silvestre - V. Sverak - A. Zlatos (2012).

Remarks

- Note that $\mathbb{A} = \mathbb{A}_0 + D$ and our theorem requires $[[\mathbb{A}]]_{\text{BMO}} \ll 1$.
- When $D = 0$, it is known that the condition $[[\mathbb{A}]]_{\text{BMO}} \ll 1$ is necessary. Ref: N.G. Meyers (1963).
- Is it necessary to require $[[D]]_{\text{BMO}} \ll 1$? Observe that $\mathbf{b} = \text{div}[D]$ and hence

$$[[D]]_{L^\infty((0,T),\text{BMO})} = \|\mathbf{b}\|_{L^\infty((0,T),\text{BMO}^{-1})}.$$

- Next goal: Study the regularity estimate for large $\mathbf{b}$. Our interesting case is $\mathbf{b} \sim \frac{1}{|x|}, \quad x \in B_1 \subset \mathbb{R}^3$.

Lorentz spaces

- For given $q > 1$, recall that

$$\|f\|_{L^q(U)} = \left\{ q \int_0^\infty s^q \left| \{(x,t) \in U : |f(x,t)| > s\} \right| \frac{ds}{s} \right\}^{1/q}.$$

- If $1 < r < \infty$, the Lorentz quasi-norm is defined by

$$\|f\|_{L^{q,r}(U)} = \left\{ q \int_0^\infty s^r \left| \{(x,t) \in U : |f(x,t)| > s\} \right|^{r/q} \frac{ds}{s} \right\}^{1/r},$$

and

$$\|f\|_{L^{q,\infty}(U)} = \sup_{s>0} s \left| \{(x,t) \in U : |f(x,t)| > s\} \right|^{1/q}.$$

- Note: $L^{q,q}(U) = L^q(U)$, and $L^{q,r_1}(U) \subset L^{q,r_2}(U)$ for all $q > 0$ and $0 < r_1 < r_2 \leq \infty$.

- Note that $\frac{1}{|x|}$ is not in $L^3(B_1)$ for $B_1 \subset \mathbb{R}^3$, but it is in $L^{3,\infty}(B_1)$.

Nonlinear C-Z theory in Lorentz spaces

Theorem (T. P. - EJDE (2017); T.P. - CJM (2017, accepted))

For given $q > 2$, $1 < r \leq \infty$ and $M > 0$, there exists $\delta = \delta(n, q, r, M) > 0$ sufficiently small such that if $[\mathbb{A}]_{\mathrm{BMO}(Q_1)} < \delta$. Then, if u is a weak solution of

$$u_t - \mathrm{div}[\mathbb{A}(x, t, u, \nabla u) + \mathbf{b}u] = \mathrm{div}[F] + f, \quad \text{in} \quad Q_2,$$

with $[[u]]_{\mathrm{BMO}(Q_1)} \leq M$, it holds that

$$\|\nabla u\|_{L^{q,r}(Q_1)} \leq C \Big[\|\nabla u\|_{L^2(Q_2)} + [[u]]_{\mathrm{BMO}} \|\mathbf{b}\|_{L^{q,r}(Q_2)} + \|F\|_{L^{q,r}(Q_2)} + \|f\|_{L^{\frac{(n+2)q}{n+4}, \frac{(n+2)r}{n+4}}(Q_2)} \Big],$$

where $C = C(n, q, r, M)$.

Local boundary regularity, and global regularity are also established.

Remarks

- In the linear setting with $F = 0$ and $f = 0$:

$$u_t - \text{div}[\mathbb{A}(x,t)\nabla u] = -\text{div}[\mathbf{b}u], \quad \text{in} \quad Q_2.$$

  The classical C-Z estimates give

$$\|\nabla u\|_{L^q(Q_1)} \leq C\left[\|\nabla u\|_{L^2(Q_2)} + \|u\|_{L^\infty(Q_2)}\|\mathbf{b}\|_{L^q(Q_2)}\right].$$

  Our estimate is

$$\|\nabla u\|_{L^{q,r}(Q_1)} \leq C\left[\|\nabla u\|_{L^2(Q_2)} + [[u]]_{\text{BMO}}\|\mathbf{b}\|_{L^{q,r}(Q_2)}\right].$$

  Hence, our theorem not only covers the singular case, but also improves the classical result to the critical borderline case.

- When $\mathbf{b} = 0$, the theorem recovers known results (E. Acerbi - G. Mingione; S.-S. Byun - L. Wang, H.-Dong-D.-Kim)

Key ideas

The condition $\mathrm{div}\mathbf{b} = 0$ is vital:

$$\int_Q u\mathbf{b} \cdot \nabla\varphi dxdt = \int_Q [u - \bar{u}_Q]\mathbf{b} \cdot \nabla\varphi dxdt.$$

Then, using Hölder's inequality and John-Nirenberg's theorem

$$\left| \int_Q u\mathbf{b} \cdot \nabla\varphi dxdt \right| \leq C(n, \alpha)[[u]]_{\mathrm{BMO}} \|\mathbf{b}\|_{L^\alpha(Q)} \|\nabla\varphi\|_{L^2(Q)}$$

for some $\alpha > 2$.

Key ideas (cont.)

- The proof is based on the ideas originated by Calderón-Zygmund (1952).
- Recent perturbation technique developed by N. Krylov (1990s, 2008); Cafferelli-Peral(1998); and Acerbi-Mingione (2007).
- My new developed techniques in the perturbation method.
- Real analysis techniques such as Vitali's covering lemma are used; stopping-time argument is employed.
- The approach should work when Ω is a manifold.

Refs: Dissipative equations

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Thank you for your attention