Phase Transition Dynamics in Geophysical Fluid Dynamics

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Outline

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I. Introduction to Dynamic Transition Theory

Basic models in

- classical, geophysical and astrophysical fluid dynamics
- statistical physics
- chemical reactions
- biological and ecological models

can all be put into dissipative dynamical systems as follows:

$$\frac{du}{dt} = L_\lambda u + G(u, \lambda) \tag{1}$$

where u is the order parameter, λ is the control parameter of the system, L_λ is a linear operator, and $G(u, \lambda)$ is the nonlinear operator.

Phase Transition

- It is a universal phenomena in most, if not all, natural systems.
- It refers to the transition of the system from one state to another, as the control parameter crosses certain critical threshold.

Unified definition of phase transitions (Ma-Wang)

Let $\beta_1(\lambda), \beta_2(\lambda), \cdots \in \mathbb{C}$ be eigenvalues of L_λ. If

$$
\begin{aligned}
&\mathrm{Re}\beta_i(\lambda) \begin{cases} < 0 & \text{if } \lambda < \lambda_0, \\ = 0 & \text{if } \lambda = \lambda_0, \\ > 0 & \text{if } \lambda > \lambda_0 \end{cases} && 1 \leq i \leq m, \\
&\mathrm{Re}\beta_j(\lambda_0) < 0 && m+1 \leq j,
\end{aligned}
$$

then the system (1) always undergoes a dynamic transition as λ crosses λ_0, and λ_0 is the critical threshold/point.

Note: The above definition was a theorem on dynamic transition we have proved; see `[Ma-Wang, Phase Transition Dynamics, 555pp, 2013, Springer]`.

This theorem ensures the validity of the above definition.

Principle of Phase Transition Dynamics (Ma-Wang):

Phase transitions of all dissipative systems can be classified into three categories: continuous, catastrophic, and random:

$\lim_{\lambda \to \lambda_0} u_\lambda = \bar{u}$	continuous transition
$\lim_{\lambda \to \lambda_0} u_\lambda \neq \bar{u}$	catastrophic transition
both $\lim_{\lambda \to \lambda_0} u_\lambda = \bar{u}$ and $\lim_{\lambda \to \lambda_0} u_\lambda \neq \bar{u}$ happen	random transition

Here $\bar{u}$ is the basic state, and u_λ are the transition states (physically, transition states correspond to local attractors).

Note:

- The above principle is ensured by the dynamic transition theorem in Phase Transition Dynamics book.
- This is a universal principle, applicable to phase transitions of all dissipative systems in Nature. It cannot be derived from classical bifurcation theory.
- It offers a guiding principle for studying phase transitions of natural systems.
- The dynamic transition theory provides a systematic approach for classifying and determining the detailed information of the transition.

II. Phase Transitions of Thermodynamical Systems

Thermodynamic Systems (Ma-Wang, 2017)

- A thermodynamic system is described by order parameters (state functions), control parameters, and thermodynamic potential, which is a functional of the order parameters.

- All thermodynamic potentials are expressed in terms of conjugate pairs. The most commonly considered conjugate thermodynamic variables are

  1) the temperature T and the entropy S, and
  2) f the generalized force and X the generalized displacement.
     Typical examples of (f, X) include (the pressure p, the volume V), (applied magnetic field H, magnetization M), (applied electric field E, electric polarization P).

Potential-Descending Principle (MW17a)

For each thermodynamic system, there are order parameters $u = (u_1, \cdots, u_N)$*, control parameters* λ*, and the thermodynamic potential functional* $F(u; \lambda)$*. For a non-equilibrium state* $u(t; u_0)$ *of the system with initial state* $u(0, u_0) = u_0$*,*

1) the potential $F(u(t; u_0); \lambda)$ *is decreasing:* $\frac{d}{dt}F(u(t; u_0); \lambda) < 0 \qquad \forall t > 0;$

2) the order parameters $u(t; u_0)$ *have a limit:* $\lim_{t\to\infty} u(t; u_0) = \bar{u};$

3) there is an open and dense set $\mathcal{O}$ *of initial data in the space of state functions, such that for any* $u_0 \in \mathcal{O}$*, the corresponding* $\bar{u}$ *is a minimum of* F*, which is called an equilibrium of the thermodynamic system:*

$$\delta F(\bar{u}; \lambda) = 0.$$

1. *The potential-descending principle leads to both the first and second laws of thermodynamics*

For the equilibrium state, PDP says that $\frac{\delta}{\delta u}F(\bar{u};\lambda)=0$, and then

$$dF(\bar{u},\lambda)=\frac{\delta}{\delta u}F(\bar{u};\lambda)\delta u+\frac{\partial F}{\partial \lambda}d\lambda=\frac{\partial F(\bar{u};\lambda)}{\partial \lambda}d\lambda,$$

which is the *first law of thermodynamics*.

For a given non-equilibrium thermodynamic state $u(t)$, the PDP tells us that

$$\frac{dF}{dt}=\frac{\delta}{\delta u}F(u(t);\lambda)\frac{du}{dt}<0 \quad \Longrightarrow \frac{\delta}{\delta u}F(u(t);\lambda)du<0.$$

Hence

$$dF(u(t),\lambda)=\frac{\delta}{\delta u}F(u(t);\lambda)du+\frac{\partial F}{\partial \lambda}d\lambda<\frac{\partial F}{\partial \lambda}d\lambda,$$

which is the *second law of thermodynamics.*

2. *PDP is a first principle of statistical mechanics.* Namely, PDP leads to all three distributions: Maxwell-Boltzmann distribution, the Bose-Einstein distribution, the Fermi-Dirac distribution.

3. Let $F(u, \lambda)$ be the thermodynamic potential of a thermodynamic system with order parameters u and control parameters λ. Then PDP gives rise to the following dynamic equation:

$$\frac{du}{dt} = -\delta F(u, \lambda). \tag{2}$$

4. *Irreversibility in Thermodynamic Systems*

- PDP offers a clear description of the irreversibility of thermodynamical systems. Consider a non-equilibrium initial state u_0, the PDP amounts to saying that the potential is decreasing:

$$\frac{d}{dt}F(u(t;u_0);\lambda) < 0 \qquad \forall t > 0.$$

  This shows that the state of the system $u(t;u_0)$ will never return to its initial state u_0 in the future time. This is exactly the irreversibility.

- Entropy S is a state function, which is the solution of basic thermodynamic equations. Thermodynamic potential is a higher level physical quantity than entropy, and consequently, is the correct physical quantity, rather than the entropy, for describing irreversibility for all thermodynamic systems.

Classical Notions of Phase Transitions:

- Ehrenfest (1933): Phase transitions are defined in terms of singularities, at the critical threshold, of such thermodynamic observable parameters as heat capacity, magnetic susceptibility, etc., which are observable.

  Classification (n-th order transition): Phase transitions are classified based on the behavior of the thermodynamic potentials, and were labeled by the n-th order derivative of the free energy that is discontinuous at the transition.

- Landau's definition (1940): The transition state of the system beaks the symmetry of the basic state $\bar{u}$. Landau's transition is of second-order.

- Topological order definition (1971, Thouless-Haldane-Kosterlitz, 2016 Nobel in Phys.): The topological structure of u_λ in the physical space differs from that of the basic state. This transition is of 3rd-order or higher-order.

Basic Theorem of Thermodynamic Phase Transitions (Ma-Wang, 2013)

- For the phase transition of a thermodynamic system, there exist only first-order, second-order and third-order phase transitions.
- Moreover the following relations between the Ehrenfest classification and the dynamical classification hold true:

$$
\begin{aligned}
&\text{second-order} && \Longleftrightarrow && \text{continuous} \\
&\text{first-order} && \longleftarrow && \text{catastrophic} \\
&\text{either first or third-order} && \longleftarrow && \text{random} \\
&\text{first-order} && \longrightarrow && \text{either catastrophic or random} \\
&\text{third-order} && \longrightarrow && \text{random with asymmetric fluctuations.}
\end{aligned}
$$

Remarks

- This theorem can only be derived using the dynamic transition theory.
- In classical thermodynamics, there is no theory to determine the Ehrenfest classification.
- In the theorem, the 1st and 2nd-order transitions on the left-hand side can only be verified and determined by experiments, while the right-hand side is rigorously determined by the dynamic transition theory.
- The 3rd-order transition cannot be determined by thermodynamic parameters, and the topological-order is sometimes used experimentally for this purpose.
- The dynamic transition theory offers an easy theoretical approach to completely determine 3rd-order transitions.

III. Dynamic Transitions in GFD and Climate Dynamics

The theory has been applied to a wide range of problems in nonlinear sciences, leading to a number of **physical predictions**:

- **Classical Fluid Dynamics:** Bénard convection, Taylor problem, and Taylor-Couette-Poiseuille flows (mechanism of the formation of the Taylor vortices)

- **Geophysical Fluid Dynamics and Climate Dynamics:** rotating Boussinesq equations (joint with C. Hsia), double-diffusive models (joint with J. Bona & C. Hsia), thermohaline circulation, ENSO (metastable states oscillation theory), ...

- **Equilibrium phase transitions**: Gas-liquid transition (the nature and theory of the critical point), ferromagnetism (asymmetry principle of fluctuations), binary systems, superconductivity, and superfluidity

- **Pattern formation and Topological Phase Transitions:**
  - Benard convection
  - Taylor-Couette-Poiseuille flows and formation of Taylor vortices
  - formation and mechanism of different patterns in Marengoni flow (with H. Dijkstra and T. Sengul),
  - quantum phase transitions – work in progress

Baroclinic instabiltiy and transitions:

The nondimensional two-layer quasi-geostrophic model (Pedlosky, 1970):

$$\text{(3)} \quad \left[\frac{\partial}{\partial t} + \frac{\partial \psi_1}{\partial x}\frac{\partial}{\partial y} - \frac{\partial \psi_1}{\partial y}\frac{\partial}{\partial x}\right]\left[\Delta\psi_1 + F(\psi_2 - \psi_1) + \beta y\right] = -r\Delta\psi_1 + \frac{1}{Re}\Delta^2\psi_1,$$

$$\text{(4)} \quad \left[\frac{\partial}{\partial t} + \frac{\partial \psi_2}{\partial x}\frac{\partial}{\partial y} - \frac{\partial \psi_2}{\partial y}\frac{\partial}{\partial x}\right]\left[\Delta\psi_2 + F(\psi_1 - \psi_2) + \beta y\right] = -r\Delta\psi_2 + \frac{1}{Re}\Delta^2\psi_2,$$

where Re is the Reynolds number, r is friction coefficient, β is the planetary vorticity factor, F is the Froude number, and the basic (shear-type) flow [Mak 85, Cai-Mak 87] is $\quad \psi_1^{(0)} = -Uy, \qquad \psi_2^{(0)} = Uy.$

Let $\psi = \frac{1}{2}\left(\psi_1 + \psi_2\right), \qquad \theta = \frac{1}{2}\left(\psi_1 - \psi_2\right) + Uy.$

Domain; $\mathcal{R} = (0, 2\pi\gamma^{-1}) \times (0, \pi)$, with γ being the wavenumber of the lowest zonal harmonic.

BC: (ψ, θ) are periodic in x (zonal), and free-slip in y (meridional).

For $2F > \gamma^2 + 1$, let $\lambda_{k,l} = \gamma^2 k^2 + l^2$, and the **critical shear** U_c be defined by

$$U_c^2 = \mathcal{U}^2(\hat{k}, \hat{l}) \stackrel{\text{def}}{=} \min_{\substack{k,l \geq 1 \\ \lambda_{k,l} < 2F}} \mathcal{U}^2(k, l),$$

$$\mathcal{U}^2(k, l) = \frac{1}{2F - \lambda_{k,l}} \left(\frac{F^2 \beta^2}{\lambda_{k,l}(F + \lambda_{k,l})^2} + \frac{\lambda_{k,l}(\frac{1}{Re}\lambda_{k,l} + r)^2}{\gamma^2 k^2} \right).$$

Transition number:

$$\boxed{b = (2F - \lambda_{\hat{k},\hat{l}})(\gamma^2 \hat{k}^2 - \hat{l}^2) + 2\hat{l}^2 \lambda_{\hat{k},\hat{l}}}$$

which captures the nonlinear interactions and dictates the types of transitions.

Theorem [Cai-Hernandez-Ong-Wang, 2017]

1. Let $2F \leq \gamma^2 + 1$. Then for system (2) and (3), the basic shear flow always stable for any shear strength U.

2. Let $2F > \gamma^2 + 1$. If the transition number $b > 0$, then the system undergoes a continuous transition to a stable periodic orbit as U crosses U_c:

$$
\begin{aligned}
\psi(x,y,t) &= \rho\gamma\hat{k}U_c \sin\left(\omega t + \gamma\hat{k}x\right)\sin\hat{l}y + O(|U-U_c|),\\
\theta(x,y,t) &= \rho\left(\frac{1}{Re}\lambda_{\hat{k},\hat{l}} + r\right)\cos\left(\omega t + \gamma\hat{k}x\right)\sin\hat{l}y\\
&\quad + \rho\frac{\gamma\hat{k}\beta F}{\lambda_{\hat{k},\hat{l}}(F+\lambda_{\hat{k},\hat{l}})}\sin\left(\omega t + \gamma\hat{k}x\right)\sin\hat{l}y + O(|U-U_c|),
\end{aligned}
\tag{5}
$$

where

$$
(6)\qquad
\begin{aligned}
\rho^2 &= \frac{4(2F-\lambda_{\hat{k},\hat{l}})(\frac{1}{Re}\lambda_{0,2\hat{l}}+r)(U-U_c)}{\gamma^2\hat{k}^2F[(2F-\lambda_{\hat{k},\hat{l}})(\gamma^2\hat{k}^2-\hat{l}^2)+2\hat{l}^2\lambda_{\hat{k},\hat{l}}](\frac{1}{Re}\lambda_{\hat{k},\hat{l}}+r)},\\
\omega &= \frac{\gamma\hat{k}\beta}{F+\lambda_{\hat{k},\hat{l}}}+O\left(|U-U_c|^{3/2}\right).
\end{aligned}
$$

3. Let $2F > \gamma^2 + 1$. If transition number $b < 0$, then the system undergoes a catastrophic transition as U crosses U_c. Also, the system bifurcates to an unstable periodic solution of the form similar to the above for $U < U_c$.

Note: If $\gamma = 1$, the transition number $b > 0$ is always positive and the system always undergoes a continuous dynamic transition leading spatiotemporal oscillations.

This suggests that a continuous transition to spatiotemporal patterns is preferable for the shear flow associated with baroclinic instability.

- The 3D continuously stratified rotating Boussinesq equations are fundamental equations in GFD;

- Dynamics associated with their basic zonal shear flows play a crucial role in understanding many important GFD processes, such as the meridional overturning oceanic circulation and the geophysical baroclinic instability;

  The linearized eigenvalue problem around the basic shear flow $u = (Uz, 0, 0)$ involves variable coefficients;

- We are developing computer-assisted method, combining the dynamic transition theory and numerical computation, to capture the stable and unstable modes, and their nonlinear interactions; see [Dijkstra-Sengul-Shen-Wang '15, Sengul-Wang '17], Marco Hernandez, Quan Wang, ...

IV. Non-Markovian Parameterizing Manifolds and Closures of SPDE

- M. D. Chekroun, H. Liu, and S. Wang: "*Approximation of Invariant Manifolds: Stochastic Manifolds for Nonlinear SPDEs I*." SpringerBriefs in Mathematics. Springer, New York, xv+127 pp., 2015.

- M. D. Chekroun, H. Liu, and S. Wang: "*Stochastic Parameterizing Manifolds and Non-Markovian Reduced Equations: Stochastic Manifolds for Nonlinear SPDEs II*." SpringerBriefs in Mathematics. Springer, New York, xvii+129 pp., 2015.

- M. D. Chekroun, H. Liu, J. McWilliams, and S. Wang: Closures for stochastic partial differential equations driven by degenerate noise, 66pp. in preparation.

The SPDEs considered

- We will be mainly concerned with SPDEs that can be written into the abstract form

$$du = (L_\lambda u + B(u,u))\,dt + dW_t, \; u \in \mathcal{H}. \tag{3}$$

- Typically, $L_\lambda = -A + P_\lambda$ where P_λ is a **bounded linear operator** depending continuously on λ (from $D(A) \subset \mathcal{H}_\alpha \subset \mathcal{H}$), and $-A$ is **sectorial**, and $\mathbf{Re}(\sigma(-A)) < 0$.
- The nonlinearity $B\colon \mathcal{H}_\alpha \times \mathcal{H}_\alpha \to \mathcal{H}$ is a bilinear mapping with $\alpha \in [0,1)$.
- Here the noise is **degenerate** and takes the form

$$W_t(\omega) = \sum_{k=1}^{N} \sigma_k W_t^k(\omega) e_k(x), \qquad t \in \mathbb{R},\; x \in \mathcal{O},\; \sigma_k \geq 0,\; \omega \in \Omega. \tag{4}$$

Our goal

Given a low-dimensional reduced phase space, e.g. $\mathcal{H}^\mathfrak{c} = \text{span}\{e_1, \cdots, e_m\}$, we aim at determining **m-dimensional closure systems** able to mimic the main (or certain) features of the SPDE dynamics. In practice, m corresponds to a **cutoff wavenumber** k_c.

Our approach

- Given a decomposition $\mathcal{H}^\mathfrak{c} \oplus \mathcal{H}^\mathfrak{s} = \mathcal{H}$, we seek for **parameterizations** of the high modes, i.e. for mapping $h : \mathcal{H}^\mathfrak{c} \to \mathcal{H}^\mathfrak{s}$ (possibly random), that will obey **key statistical constraints** with respect to the **ergodic invariant measure** when it exists (Hairer, Mattingly,...).

In a first attempt, let us look for **deterministic high-mode parameterizations** of the form

$$h(\xi) := \sum_{n=m+1}^{\infty} \Phi_n(\xi) e_n, \ \ \xi \in \mathcal{H}^c.$$

Guidance from the ergodic theory of SPDEs (Hairer, Mattingly, Flandoli,...)

Assume that the SPDE (3) admits an (unique) **ergodic invariant measure** μ. Let us introduce the **normalized parameterization defect** (over $[0, T]$) associated with Φ_n, namely

$$\mathcal{Q}_n(T;\omega) = \frac{\int_0^T \left| \langle u_s(t;\omega), e_n \rangle - \Phi_n(u_c(t;\omega)) \right|^2 \mathrm{d}\, t}{\int_0^T |\langle u_s(t;\omega), e_n\rangle|^2 \,\mathrm{d}\, t}, \ \omega \in \Omega.$$

Then for each $n \geq m+1$, and "any" solution u of Eq. (3), $\lim_{T\to\infty} \mathcal{Q}_n(T;\omega)$ exists $\mathbb{P}$-a.s., and

$$\lim_{T\to\infty} \mathcal{Q}_n(T;\omega) = C_n \int_{(\xi,\zeta)\in\mathcal{H}^c\times\mathcal{H}^s} |\langle \zeta, e_n\rangle - \Phi_n(\xi)|^2 \,\mathrm{d}\mu \ , \quad \mathbb{P}\text{-a.s.},$$

where

$$C_n = \Big(\int_{(\xi,\zeta)\in\mathcal{H}^c\times\mathcal{H}^s} |\langle \zeta, e_n\rangle|^2 \,\mathrm{d}\mu \Big)^{-1}.$$

Variational perspective

- Intuitively **one wants $\lim_{T\to\infty} \mathcal{Q}_n(T;\omega) < 1$ $\mathbb{P}$-a.s.** for as much as possible e_n-modes, $n \geq m+1 \implies$ one wants a **parameterizing manifold (PM)**.
- The extreme case in which for all $n \geq m+1$, **$\mathcal{Q}_n = 1$, corresponds to the Galerkin approximation** (i.e. no high-mode parameterizations) whereas the case **$\mathcal{Q}_n = 0$ (for all n) includes inertial manifolds** or other **slow manifolds**, when they exist!
- One are thus naturally inclined to look for solutions to the following sequence of variational problems

$$\min_{\Phi_n} \int_{(\xi,\zeta)\in\mathcal{H}^{\mathfrak{c}}\times\mathcal{H}^{\mathfrak{s}}} |\langle \zeta, e_n\rangle - \Phi_n(\xi)|^2 \,\mathrm{d}\mu, \;\; n \geq m+1.$$

- One can prove by using the general disintegration theorem of probability measures that the optimal parameterization (when Φ_n is taken in $L^2_{\mu_{\mathfrak{c}}}(\mathcal{H}^{\mathfrak{c}},\mathcal{H}^{\mathfrak{s}})$)

$$\boxed{h^*(\xi) = \int_{\mathcal{H}^{\mathfrak{s}}} \zeta \,\mathrm{d}\mu_\xi(\zeta), \;\; \xi \in \mathcal{H}^{\mathfrak{c}},}$$

denotes the **optimal Markovian high-mode parameterization**. Here, μ_ξ denotes the **disintegrated probability measure** on the high-mode space $\mathcal{H}^{\mathfrak{s}}$ and that is conditioned on the low-mode variable ξ in $\mathcal{H}^{\mathfrak{c}}$.
- Recall that

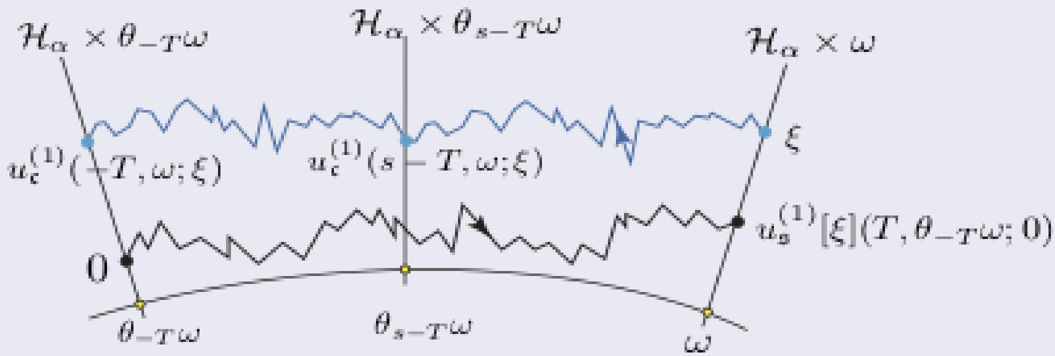
$$\mu(B\times F) = \int_F \mu_\xi(B)\,\mathrm{d}\mu_{\mathfrak{c}}(\xi), \;\; B\times F \in \mathcal{B}(\mathcal{H}^{\mathfrak{s}})\otimes\mathcal{B}(\mathcal{H}^{\mathfrak{c}})$$

where $\mu_{\mathfrak{c}}$ is the *push-forward* of the measure μ by $\Pi_{\mathfrak{c}}$, *i.e.* $\mu_{\mathfrak{c}}(F) = \mu(\Pi_{\mathfrak{c}}^{-1}(F))$.

Mode-dependent minimization problems

With this purpose in mind, given a reduced phase space $\mathcal{H}^{\mathfrak{c}} := \mathrm{span}\{e_1, \cdots, e_m\}$, and a fully resolved solution $u(t,\omega)$ of (3) available over a training interval $[0,T]$ for a noise realization ω, we propose to solve $n \geq m+1$ the following **multiscale minimization problem**

$$
\begin{cases}
\min\limits_{\tau} \displaystyle\int_0^T \left| u_n(t,\omega) - u_n^{(1)}[u_{\mathfrak{c}}(t,\omega)](\tau, \theta_{-\tau}\omega) \right|^2 \mathrm{d}t, & \text{(5a)} \\
\text{where } u_n^{(1)}[\xi](\tau, \theta_{-\tau}\omega) \text{ is the sol., at } t=0 \text{ and for the } \textbf{noise realization } \boldsymbol{\omega}\text{, of} & \\
\mathrm{d}u_{\mathfrak{c}}^{(1)} = L_{\mathfrak{c}} u_{\mathfrak{c}}^{(1)}(s)\,\mathrm{d}s + \mathrm{d}\Pi_{\mathfrak{c}} W_s, \quad s \in [-\tau, 0], & \text{(5b)} \\
\mathrm{d}u_n^{(1)} = \left[\beta_n u_n^{(1)}(s) + \Pi_n B\big(u_{\mathfrak{c}}^{(1)}(s-\tau), u_{\mathfrak{c}}^{(1)}(s-\tau)\big)\right]\mathrm{d}s + \mathrm{d}\Pi_n W_{s-\tau}, \ \ s \in [0,\tau], & \text{(5c)} \\
\text{with } u_{\mathfrak{c}}^{(1)}(s,\omega)|_{s=0} = \xi \in \mathcal{H}^{\mathfrak{c}}, \text{ and } u_n^{(1)}(s, \theta_{-\tau}\omega)|_{s=0} = 0. & \text{(5d)}
\end{cases}
$$



Theorem (M. Chekroun, H. Liu, J. McWilliams, & S. W., 2017)

Consider the SPDE (3) with L being self-adjoint and the noise being given by

$$W_t(\omega) = \sum_{k=1}^{N} \sigma_k W_t^k(\omega) e_k, \;\; \sigma_k \geq 0, \; \omega \in \Omega. \tag{8}$$

Then given a reduced phase space $\mathcal{H}^{\mathfrak{c}}$, the stochastic process $(u_{\mathfrak{c}}^{(1)}, u_n^{(1)})$ evolving in $\mathcal{H}^{\mathfrak{c}} \times \mathrm{span}\{e_n\}$ and solving the **backward-forward system** (5b)-(5c) subject to the boundary conditions (5d), has the following integral representation,

$$u_n^{(1)}[\xi](\tau, \theta_{-\tau}\omega) = -e^{\beta_n \tau} \Pi_n W_{-\tau}(\omega) + \beta_n \int_{-\tau}^{0} e^{-\beta_n s} \Pi_n W_s(\omega) \, \mathrm{d}s$$
$$+ \int_{-\tau}^{0} e^{-\beta_n s} \Pi_n B\big(u_{\mathfrak{c}}^{(1)}(s, \omega; \xi), u_{\mathfrak{c}}^{(1)}(s, \omega; \xi)\big) \, \mathrm{d}s, \;\; \xi \in \mathcal{H}^{\mathfrak{c}},$$

and

$$u_{\mathfrak{c}}^{(1)}(s, \omega; \xi) = e^{sL_{\mathfrak{c}}} \xi - \int_{\tau}^{0} e^{(s-s')L_{\mathfrak{c}}} L_{\mathfrak{c}} \Pi_{\mathfrak{c}} W_{s'}(\omega) \, \mathrm{d}s' + \Pi_{\mathfrak{c}} W_{\tau}(\omega), \qquad s \in [-\tau, 0].$$

Moreover, the **stochastic process** $u_n^{(1)}[\xi](\tau, \theta_{-\tau}\omega; 0)$ admits the following expression

$$u_n^{(1)}[\xi](\tau, \theta_{-\tau}\omega) = \sigma_n e^{\tau \beta_n} W_{-\tau}^n(\omega) + Z_\tau^n(\boldsymbol{\beta}, \omega) + \sum_{i_1=1}^{m} \sum_{i_2=1}^{m} \Big(P_\tau^{n, i_1, i_2}(\boldsymbol{\beta}, \omega)$$
$$+ C_\tau^{n, i_1, i_2}(\boldsymbol{\beta}, \omega) \xi_{i_1} + C_\tau^{n, i_2, i_1}(\boldsymbol{\beta}, \omega) \xi_{i_2} + D_{i_1, i_2}^n(\tau, \boldsymbol{\beta}) \xi_{i_1} \xi_{i_2} \Big) \langle B(\boldsymbol{e}_{i_1}, \boldsymbol{e}_{i_2}), \boldsymbol{e}_n \rangle.$$

Explicit formulas

Here $\sigma_n = 0$ and $Z^n_\tau(\boldsymbol{\beta},\omega) = 0$ if $n > N$, and where for all n,

$$D^n_{i_1,i_2}(\tau,\boldsymbol{\beta}) = \begin{cases} \frac{1-\exp\left(-(\beta_{i_1}+\beta_{i_2}-\beta_n)\tau\right)}{\beta_{i_1}+\beta_{i_2}-\beta_n}, & \text{if } \beta_{i_1}+\beta_{i_2}-\beta_n \neq 0, \\ \tau, & \text{otherwise} \end{cases} \tag{9}$$

while the **path-dependent coefficients** are given by $Z^n_\tau(\boldsymbol{\beta},\omega) := \sigma_n \beta_n \int_{-\tau}^0 e^{-s\beta_n} W^n_s(\omega)\,\mathrm{d}s$, and

$$P^{n,i_1,i_2}_\tau(\boldsymbol{\beta},\omega) = \sum_{j=1}^{4} M^{n,i_1,i_2}_{j,\tau}(\omega), \;\; C^{n,i_1,i_2}_\tau(\boldsymbol{\beta},\omega) = \sum_{j=5}^{6} M^{n,i_1,i_2}_{j}(\omega), \quad \text{with}$$

$$\begin{aligned}
M^{n,i_1,i_2}_{1,\tau}(\boldsymbol{\beta},\omega) &:= \sigma_{i_1}\sigma_{i_2}\int_{-\tau}^0 e^{-\beta_n s} W^{i_1}_s(\omega) W^{i_2}_s(\omega)\,\mathrm{d}s, \\
M^{n,i_1,i_2}_{2,\tau}(\boldsymbol{\beta},\omega) &:= -\sigma_{i_1}\sigma_{i_2}\beta_{i_2}\int_{-\tau}^0 \left(e^{-\beta_n s} W^{i_1}_s(\omega)\int_s^0 e^{(s-s')\beta_{i_2}} W^{i_2}_{s'}(\omega)\,\mathrm{d}s'\right)\mathrm{d}s, \\
M^{n,i_1,i_2}_{3,\tau}(\boldsymbol{\beta},\omega) &:= M^{n,i_2,i_1}_{2,\tau}(\boldsymbol{\beta},\omega), \;\; M^{n,i_1,i_2}_{5,\tau}(\boldsymbol{\beta},\omega) := \sigma_{i_2}\int_{-\tau}^0 e^{(\beta_{i_1}-\beta_n)s} W^{i_2}_s(\omega)\,\mathrm{d}s, \\
M^{n,i_1,i_2}_{4,\tau}(\boldsymbol{\beta},\omega) &:= \sigma_{i_1}\sigma_{i_2}\beta_{i_1}\beta_{i_2}\int_{-\tau}^0 \left(e^{-\beta_n s}\int_s^0 e^{(s-s')\beta_{i_1}} W^{i_1}_{s'}(\omega)\,\mathrm{d}s'\int_s^0 e^{(s-s')\beta_{i_2}} W^{i_2}_{s'}(\omega)\,\mathrm{d}s'\right)\mathrm{d}s, \\
M^{n,i_1,i_2}_{6,\tau}(\boldsymbol{\beta},\omega) &:= -\sigma_{i_2}\beta_{i_2}\int_{-\tau}^0 \left(e^{(\beta_{i_1}-\beta_n)s}\int_s^0 e^{(s-s')\beta_{i_2}} W^{i_2}_{s'}(\omega)\,\mathrm{d}s'\right)\mathrm{d}s.
\end{aligned} \tag{10}$$

- These **non-Markovian M_n-terms** solve **auxiliary Random Auxiliary Equations (RDEs)**.

Application to a stochastic Burgers-Sivashinsky equation

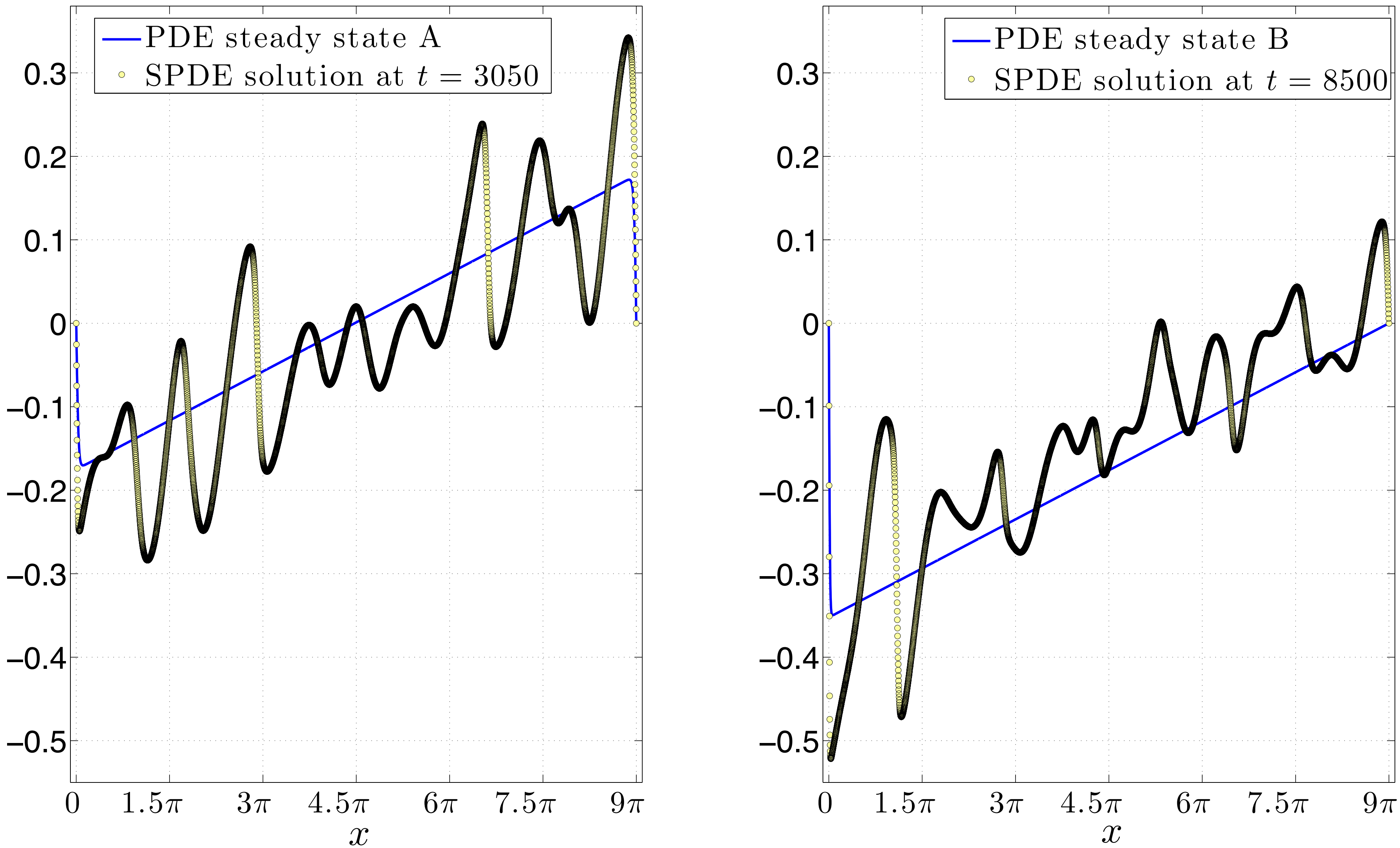
We consider the Burgers-Sivashinsky equation driven by a **degenerate** additive noise:

$$
\begin{aligned}
&\mathrm{d}u = \big(\nu u_{xx} + \lambda u - u u_x\big)\mathrm{d}t + \mathrm{d}W_t(x,\omega), && (x,t) \in (0,l) \times \mathbb{R}^+, \\
&u(0,t;\omega) = u(l,t;\omega) = 0, && t \geq 0, \\
&u(x,0;\omega) = u_0(x), && x \in (0,l).
\end{aligned}
$$

with $W_t(x,\omega) = \sum_{k=1}^{N} \sigma_k W_t^k(\omega) e_k(x)$, $\sigma_k \geq 0$, $N \in \mathbb{Z}^+$.

The SPDE is set in a parameter regime where **sharp spatial gradient** is present in the spatiotemporal field:

- $l = 9\pi$, $\nu = 0.01$, $\lambda = (100+1)\nu\pi^2/L^2$. We have **10 unstable eigenmodes**.
- The eigenmodes $15 - 30$ are forced, $\sigma_k = 4\sqrt{\nu}\, k^{-2}/\big(\sum_{k=k_s}^{k_f} k^{-2}\big)$.



PDE steady state A
SPDE solution at $t = 3050$
0.3
0.2
0.1
0
−0.1
−0.2
−0.3
−0.4
−0.5
0
1.5π
3π
4.5π
6π
7.5π
9π
x
PDE steady state B
SPDE solution at $t = 8500$

PDF and PSD of the TV-norm anomalies, $\|u\|_{\mathrm{TV}} - \langle \|u\|_{\mathrm{TV}} \rangle$