Kinetic Theory for Geophysical Flows

Corentin Herbert
CNRS, ENS de Lyon, France (GMT+1)

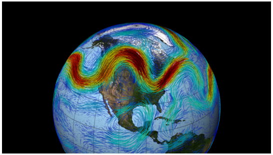
November 20, 2017 — Banff International Research Station (GMT-7)

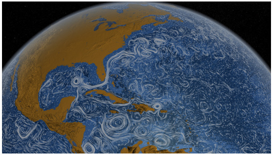
In collaboration with:

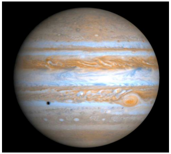
F. Bouchet ENS Lyon, France
G. Falkovich Weizmann Institute, Israel
A. Frishman Princeton University, USA

Large-scale structures in geophysical flows

Geophysical flows self-organize into large-scale coherent structures, which play a major part in weather/climate.

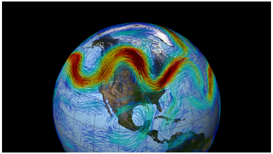
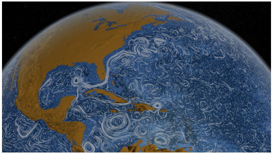
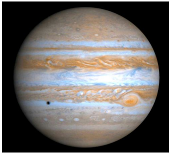




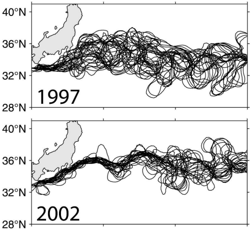


Large-scale structures in geophysical flows

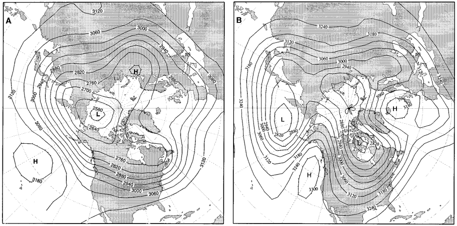
Geophysical flows self-organize into large-scale coherent structures, which play a major part in weather/climate.

They fluctuate and undergo abrupt transitions.



Kuroshio path[1]



Zonal/blocked Jet Stream transition[2]

[1] B Qiu and S. M. Chen (2005). *J. Phys. Oceanogr.*

[2] E. R. Weeks et al. (1997). *Science*

Turbulent flows and degrees of freedom

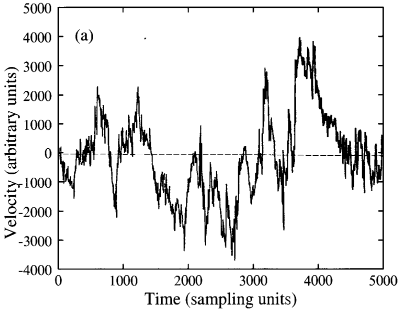
Navier-Stokes equations: nonlinear term couples wide range of scales.

$$\partial_t \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{u} = -\frac{1}{\rho} \nabla P + \nu \Delta \mathbf{u}, \qquad \mathrm{Re} = UL/\nu, \qquad \# DOF \sim \mathrm{Re}^{9/4} \approx 10^{20}$$

Turbulent flows and degrees of freedom

Navier-Stokes equations: nonlinear term couples wide range of scales.

$$\partial_t \mathbf{u} + \mathbf{u} \cdot \boldsymbol{\nabla} \mathbf{u} = -\frac{1}{\rho} \boldsymbol{\nabla} P + \nu \Delta \mathbf{u}, \qquad \mathrm{Re} = UL/\nu, \qquad \# DOF \sim \mathrm{Re}^{9/4} \approx 10^{20}$$

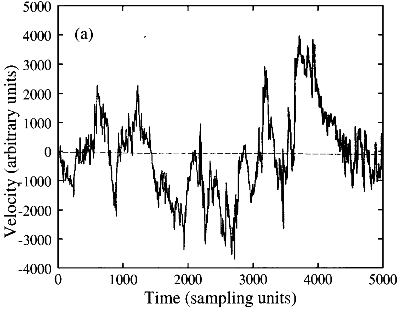


ONERA wind tunnel (Gagne, Hopfinger)

Turbulent flows and degrees of freedom

Navier-Stokes equations: nonlinear term couples wide range of scales.

$$\partial_t \mathbf{u} + \mathbf{u}\cdot\boldsymbol{\nabla}\mathbf{u} = -\frac{1}{\rho}\boldsymbol{\nabla}P + \nu\Delta\mathbf{u}, \qquad \mathrm{Re} = UL/\nu, \qquad \#DOF \sim \mathrm{Re}^{9/4} \approx 10^{20}$$



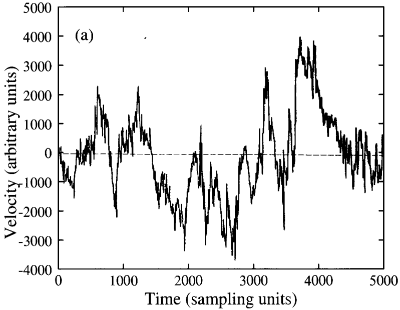
ONERA wind tunnel (Gagne, Hopfinger)

Statistical Physics Approach

Turbulent flows and degrees of freedom

Navier-Stokes equations: nonlinear term couples wide range of scales.

$$\partial_t \mathbf{u} + \mathbf{u}\cdot\nabla\mathbf{u} = -\frac{1}{\rho}\nabla P + \nu\Delta\mathbf{u}, \qquad \mathrm{Re} = UL/\nu, \qquad \#DOF \sim \mathrm{Re}^{9/4} \approx 10^{20}$$



ONERA wind tunnel (Gagne, Hopfinger)

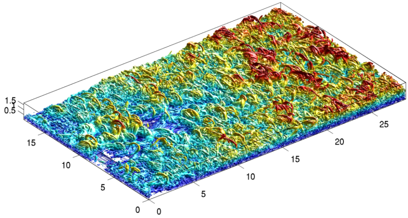
Statistical Physics Approach

Difficult problem: closure !

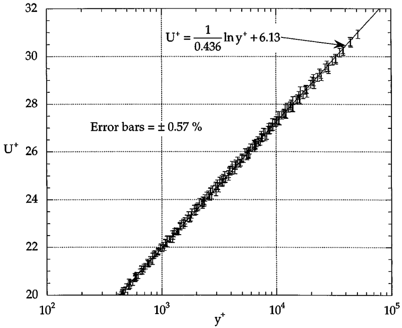
Mean-flow/turbulence interactions

How to compute velocity profile in turbulent shear flows?

There is no consistent theory of mean-flow/turbulence interactions.



E.g. Turbulent Boundary Layer

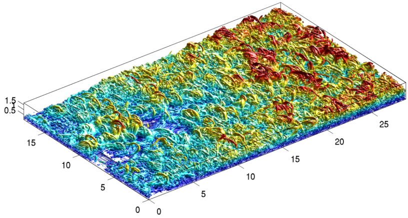


Experimental data for pipe flow[3]

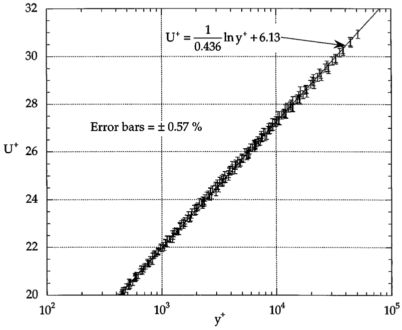
[3] M. V. Zagarola and A. J. Smits (1997). *Phys. Rev. Lett.*

Mean-flow/turbulence interactions

How to compute velocity profile in turbulent shear flows?
There is no consistent theory of mean-flow/turbulence interactions.



E.g. Turbulent Boundary Layer



Experimental data for pipe flow[3]

Conservation equations: $\varepsilon \approx -\tau U'$ with $\tau \equiv \langle uv \rangle \approx$ const.

- 3D: The mean-flow feeds turbulence, $\tau U' < 0$. *Not closed! $\varepsilon(y)$?*
- 2D: Turbulence feeds the mean-flow, $\tau U' > 0$.

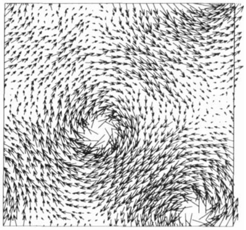
[3] M. V. Zagarola and A. J. Smits (1997). *Phys. Rev. Lett.*

Finite-size effects in 2D turbulence: condensation

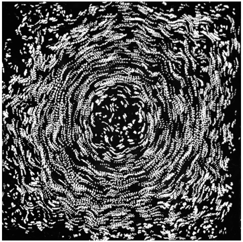
Simplest possible system: 2D Navier-Stokes with linear friction

$$\partial_t \omega + \mathbf{u} \cdot \nabla \omega = \nu \Delta \omega - \alpha \omega + f_\omega .$$

When α is small enough, the inverse cascade reaches the box scale: "condensation".



DNS[4]



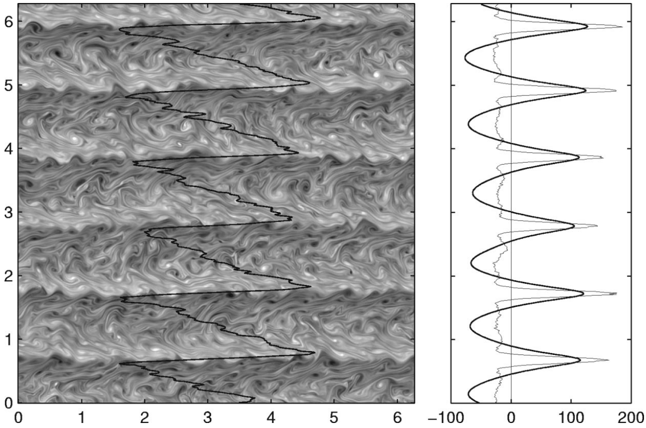
ANU 2D experiments (EM, Faraday waves)[5]

[4] L. M. Smith and V. Yakhot (1994). *J. Fluid Mech.*

[5] H Xia et al. (2009). *Phys. Fluids*; see also J. Sommeria (1986). *J. Fluid Mech.*

Zonal Jets in Rotating Flows

Beta-plane turbulence: $\partial_t \omega + \mathbf{u} \cdot \nabla \omega + \beta v = \nu \Delta \omega - \alpha \omega + f_\omega .$



Potential vorticity and zonal velocity profile[6].

[6] S. Danilov and D. Gurarie (2004). *Phys. Fluids*; see also P. B. Rhines (1975). *J. Fluid Mech.* M. E. Maltrud and G. K. Vallis (1991). *J. Fluid Mech.* B Galperin and S Sukoriansky (2001). *Phys. Fluids*,...

Main Questions and Theoretical Tools

Generic questions:

- Can we predict self-organization of geophysical flows into large scale coherent structures?
- Characterize the attractors of geophysical turbulence
- Study fluctuations around the mean state
- What aspects of abrupt transitions in turbulent flows are predictable?

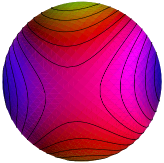
Tools from Statistical Physics

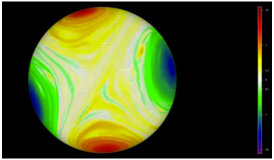
Main Questions and Theoretical Tools

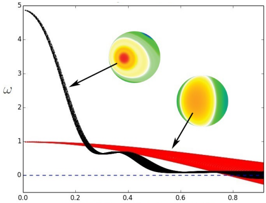
Generic questions:

- Can we predict self-organization of geophysical flows into large scale coherent structures?
- Characterize the attractors of geophysical turbulence
- Study fluctuations around the mean state
- What aspects of abrupt transitions in turbulent flows are predictable?

Equilibrium States[7] vs. numerical simulations[8]:







The set of MRS equilibria is huge: difficult to make quantitative predictions.
Can we find a theory which provides quantitative predictions based on forcing/dissipation?

[7] C. Herbert (2013). *J. Stat. Phys.*
[8] W. Qi and J. B. Marston (2014). *J. Stat. Mech.*

Main Questions and Theoretical Tools

Generic questions:

- Can we predict self-organization of geophysical flows into large scale coherent structures?
- Characterize the attractors of geophysical turbulence
- Study fluctuations around the mean state
- What aspects of abrupt transitions in turbulent flows are predictable?

Tools from Non-Equilibrium Statistical Physics

- Kinetic Theory
- Large Deviation Theory

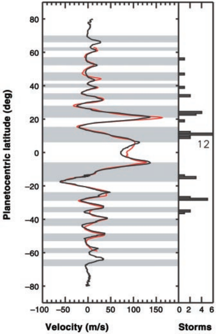
Outline

1. Introduction
2. Kinetic Theory: timescale separation
3. Testing Kinetic Theory: mean-flow and fluctuations
4. Conclusion

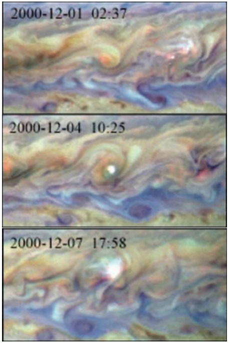
Timescale separation in geophysical flows

In some flows, there is a *natural timescale separation*, usually associated to a broken symmetry of the Navier-Stokes equations.

E.g. Jupiter[7]:



Zonal wind measured by Voyager 2 (1979, red) and Cassini (2000, black).



Cassini

[7] C. C. Porco et al. (2003). *Science*.

Adiabatic elimination of fast variables[8] (*stochastic averaging*)

Slow-fast SDE:

$$dX_t = f(X_t, Y_t)dt + \sqrt{2\epsilon}dW_t,$$
$$dY_t = \alpha^{-1}g(X_t, Y_t)dt + \sqrt{\alpha^{-1}}h(X_t, Y_t)dW_t.$$

- Joint PDF $P(x, y; t)$; Fokker-Planck equation $\partial_t P = (\alpha^{-1}L_0 + L_1)P$.
- Stationary distribution for fast modes at fixed x and projection operator:

$$L_0 P_\infty^x(y) = 0, \quad \mathcal{P}\phi = P_\infty^x(y) \int dy \phi(x, y).$$

- Write $P_s = \mathcal{P}P$, $P_f = (1 - \mathcal{P})P$. We have $\partial_t P_s = \mathcal{P}(\alpha^{-1}L_0 + L_1)P = \mathcal{P}L_1 P$.
- At lowest order, $\partial_t P_s = \mathcal{P}L_1 P_s + O(\alpha)$ and $P_s(x, y) = P_\infty^x(y)Q(x)$ with

$$\frac{\partial Q}{\partial t} = -\frac{\partial}{\partial x}\left[\mathbb{E}_\infty^x[f]Q(x)\right] + \epsilon\frac{\partial^2}{\partial x^2}Q + O(\alpha).$$

Finally, after adiabatic reduction:

$$dX_t = \mathbb{E}_\infty^{X_t}[f]dt + \sqrt{2\epsilon}dW_t.$$

[8] e.g. C. W. Gardiner (2009). *Handbook of Stochastic Methods for physics, chemistry, and the natural sciences*. 4th edition. Springer, Berlin.

Adiabatic elimination of fast variables: zonal jets

Reynolds decomposition for the zonal jets

$\omega = \bar{\omega} + \omega'$, with $\bar{\cdot}$ the projection on the (slow) zonal modes. Formally,

$$\partial_t \bar{\omega} + L_{\bar{\omega}}[\bar{\omega}] = -\partial^i \overline{u_i' \omega'} + \bar{\eta},$$
$$\partial_t \omega' + L'_{\bar{\omega}}[\omega'] = -u_i' \partial^i \omega' + \partial^i \overline{u_i' \omega'} + \eta'.$$

Adiabatic reduction at lowest order[9]**:**

$$\partial_t \bar{\omega} + L_{\bar{\omega}}[\bar{\omega}] = -\partial^i \mathbb{E}_{\bar{\omega}}[\overline{u_i' \omega'}] + \bar{\eta},$$
$$\partial_t \omega' + L'_{\bar{\omega}}[\omega'] = \eta'.$$

- No UV divergences
- Eddy-eddy interactions do not contribute at leading order.

The fluctating vorticity field is an Ornstein-Uhlenbeck process characterized by the two-point correlation function $g(\mathbf{r}_1, \mathbf{r}_2, t) = \mathbb{E}_{\bar{\omega}}[\omega'(\mathbf{r}_1, t)\omega'(\mathbf{r}_2, t)]$, which satisfies the Lyapunov equation:

$$\partial_t g + {L'}^{(1)}_{\bar{\omega}} g + {L'}^{(2)}_{\bar{\omega}} g = C',$$

with $C'(\mathbf{r}_1, \mathbf{r}_2, t) = \mathbb{E}[\eta'(\mathbf{r}_1, t)\eta'(\mathbf{r}_2, t)]$ the correlation matrix of the Gaussian white noise η'.

[9] F. Bouchet et al. (2013). *J. Stat. Phys.*

Adiabatic elimination of fast variables: zonal jets

Reynolds decomposition for the zonal jets

$\omega = \bar{\omega} + \omega'$, with $\bar{\cdot}$ the projection on the (slow) zonal modes.
Formally,

$$\partial_t \bar{\omega} + L_{\bar{\omega}}[\bar{\omega}] = -\partial^i \overline{u_i' \omega'} + \bar{\eta},$$
$$\partial_t \omega' + L'_{\bar{\omega}}[\omega'] = -u_i' \partial^i \omega' + \partial^i \overline{u_i' \omega'} + \eta'.$$

Adiabatic reduction at lowest order:

$$\partial_t \bar{\omega} + L_{\bar{\omega}}[\bar{\omega}] = -\partial^i \mathbb{E}_{\bar{\omega}}[\overline{u_i' \omega'}] + \bar{\eta},$$
$$\partial_t \omega' + L'_{\bar{\omega}}[\omega'] = \eta'.$$

- No UV divergences
- Eddy-eddy interactions do not contribute at leading order.

Numerical simulations in the quasi-linear framework[9]:

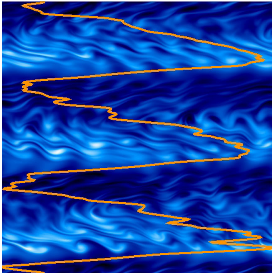
- Stochastic Structural Stability Theory[10]
- Cumulant Expansion "CE2"[11]

[9] T. Schneider and C. C. Walker (2006). *J. Atmos. Sci.* P. A. O'Gorman and T. Schneider (2007). *Geophys. Res. Lett.* K. Srinivasan and W. R. Young (2012). *J. Atmos. Sci.*

[10] B. F. Farrell and P. J. Ioannou (2003). *J. Atmos. Sci.* B. F. Farrell and P. J. Ioannou (2007). *J. Atmos. Sci.*

[11] S. M. Tobias and J. B. Marston (2013). *Phys. Rev. Lett.* J. B. Marston et al. (2016). *Phys. Rev. Lett.*

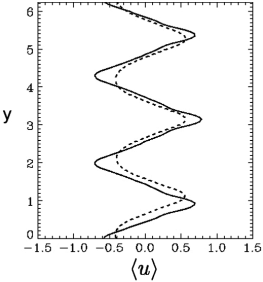
Comparing QL numerical simulations with DNS



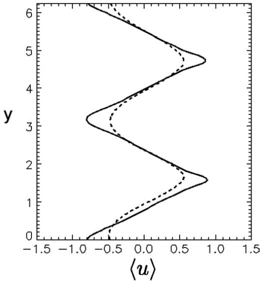
Cumulant Expansion CE2 for barotropic zonal jets[12]

$$R_\beta = \sqrt{\frac{U\beta^{1/5}}{2\varepsilon^{2/5}}}$$

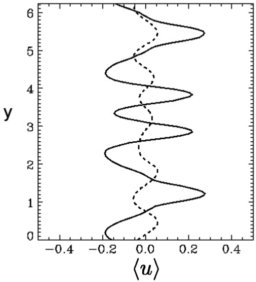
$R_\beta = 2.12$



$R_\beta = 1.98$



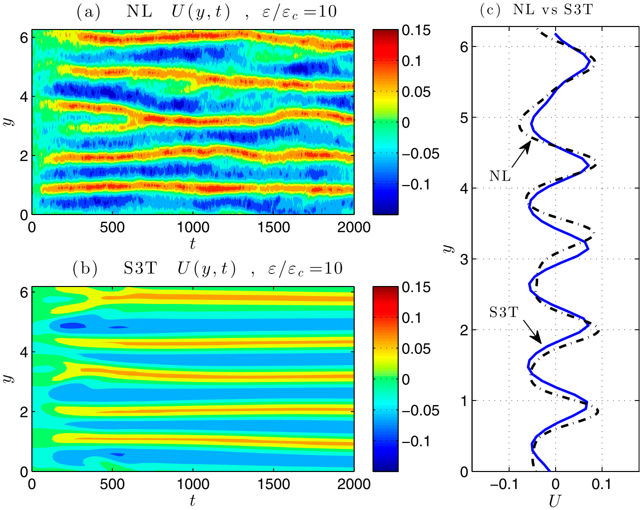
$R_\beta = 1.24$



[12] S. M. Tobias and J. B. Marston (2013). *Phys. Rev. Lett.*

Comparing QL numerical simulations with DNS

Stochastic Structural Stability Theory[13]



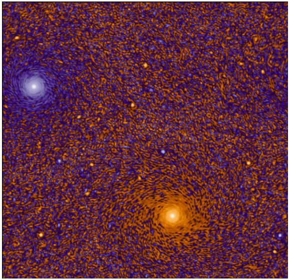
[13] N. C. Constantinou et al. (2014). *J. Atmos. Sci.*

Outline

1. Introduction
2. Kinetic Theory: timescale separation
3. Testing Kinetic Theory: mean-flow and fluctuations
4. Conclusion

Explicit computations in the vortex condensate

Let us go back to 2D Navier-Stokes in a periodic square box with linear friction and small-scale random forcing: $\partial_t \omega + \mathbf{u} \cdot \nabla \omega = \nu \Delta \omega - \alpha \omega + f_\omega$.

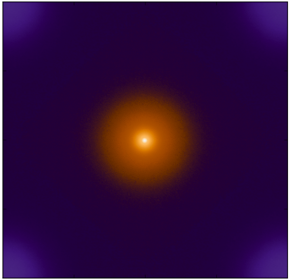


DNS: 1024^2, $k_f = 100$, hyperviscosity, $\alpha = 1.1 \times 10^{-4}$.

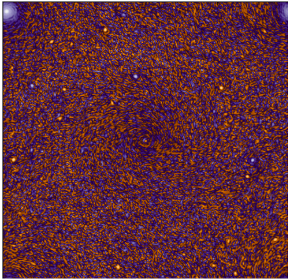
Explicit computations in the vortex condensate

Reynolds decomposition:

$$\mathbf{v} = \mathbf{U} + \mathbf{u},\ \text{with } \mathbf{U} = U\mathbf{e}_\theta, \mathbf{u} = u\mathbf{e}_\theta + v\mathbf{e}_r \text{ and } \langle \mathbf{u} \rangle = 0,$$
$$\omega = \Omega + \omega',\ \text{with } \langle \omega' \rangle = 0.$$
$$\partial_t \Omega + \mathbf{U} \cdot \nabla \Omega = -\alpha \Omega - \nabla \cdot \langle \mathbf{u}\omega' \rangle.$$



Ω



ω'

DNS: 1024^2, $k_f = 100$, hyperviscosity, $\alpha = 1.1 \times 10^{-4}$.

Explicit computations in the vortex condensate

Reynolds decomposition:

$$\mathbf{v} = \mathbf{U} + \mathbf{u}, \text{ with } \mathbf{U} = U\mathbf{e}_\theta, \mathbf{u} = u\mathbf{e}_\theta + v\mathbf{e}_r \text{ and } \langle \mathbf{u} \rangle = 0,$$
$$\omega = \Omega + \omega', \text{ with } \langle \omega' \rangle = 0.$$
$$\partial_t \Omega + \mathbf{U} \cdot \nabla \Omega = -\alpha \Omega - \nabla \cdot \langle \mathbf{u} \omega' \rangle.$$

Timescale separation

Perturbative expansion of the equations of motion in $\delta = \alpha L^{2/3}/\varepsilon^{1/3} \ll 1$ leads at first order to (Momentum and energy balance)[14]:

$$r^{-1}\partial_r(r^2 \langle uv \rangle) = -\alpha r U,$$
$$r^{-1}\partial_r(rU\langle uv \rangle) + \alpha U^2 = \varepsilon.$$

Solution:

$$U = \sqrt{3\varepsilon/\alpha}, \qquad \langle uv \rangle = -r\sqrt{\alpha\varepsilon/3}.$$

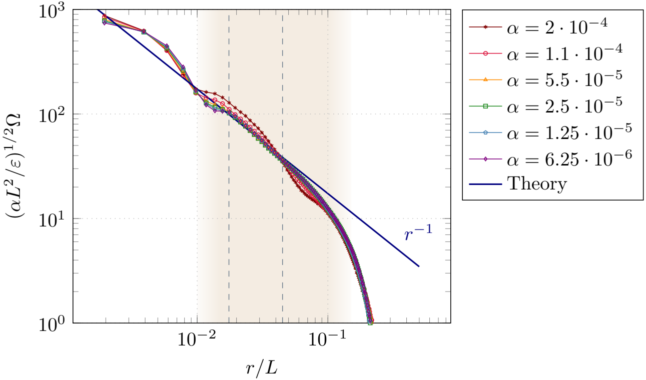
Therefore $\Omega(r) = \sqrt{3\varepsilon/\alpha}\, r^{-1}$.

Global energy balance neglecting small-scale dissipation yields $U_{\text{rms}} = \sqrt{\varepsilon/\alpha}$.

[14] J. Laurie et al. (2014). *Phys. Rev. Lett.*

Explicit computation for the mean vorticity profile

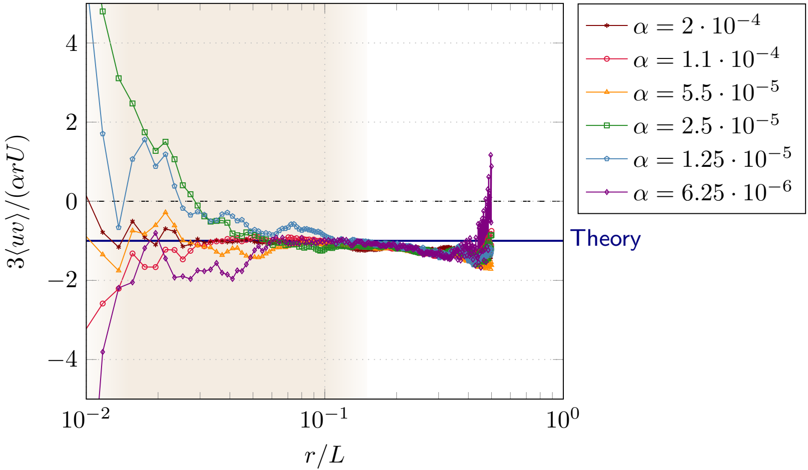
Theoretical prediction: $\Omega(r) = \sqrt{3\varepsilon/\alpha}\, r^{-1} = \sqrt{3}(\varepsilon L)^{1/3}\delta^{-1/2} r^{-1}$.



Our DNS (512^2 and 1024^2) support the α-scaling on a wide range of α, and seem compatible with the r-scaling.

Explicit computation for the Reynolds tensor[15]

$\langle uv \rangle / U^2 = O(\delta^{3/2})$ and not sign definite.

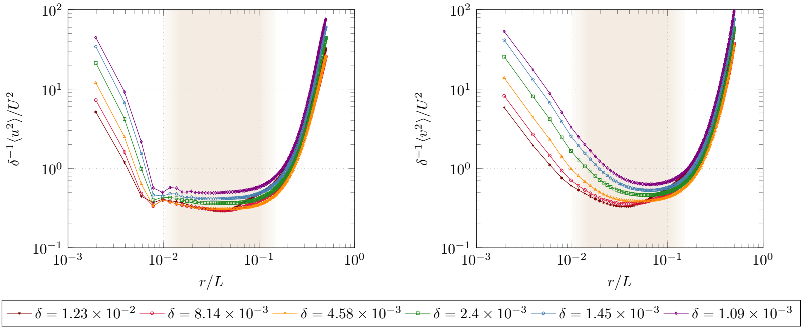


DNS: 512^2, $k_F = 100$, hyperviscosity, $\sim$ 300000 turnover times.

[15] A. Frishman and C. Herbert (submitted). *Phys. Rev. Lett.*

Fluctuations: turbulent energy profile

DNS: 512^2, $k_F = 100$, hyperviscosity.



In the region of interest:

- $\langle u^2 + v^2 \rangle \ll U^2$ confirmed.
- Weak dependence on r.

Turbulent energy profile not given by leading order energy/momentum balance.

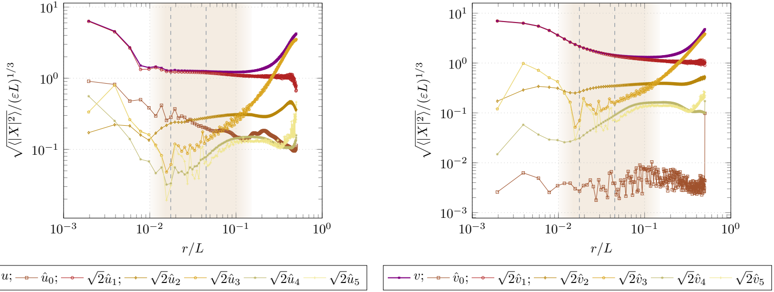
Fluctuations: turbulent energy profile

Quasi-linear dynamics:

$$\partial_t \omega + L_U[\omega] = \eta, \quad L_U = \frac{U}{r}\partial_\theta - \frac{\Omega'}{r}\partial_\theta \Delta^{-1} + \alpha\,\mathrm{Id} - \nu\Delta.$$

Two-point correlation function: $g(\mathbf{r}_1, \mathbf{r}_2) = \langle \omega(\mathbf{r}_1)\omega(\mathbf{r}_2)\rangle$.
Lyapunov equation: $\partial_t g + [L_U^{(1)} + L_U^{(2)}]g = C, \quad C = \langle \eta(\mathbf{r}_1)\eta(\mathbf{r}_2)\rangle$.

Decompose into azimuthal harmonics $u(r,\theta) = \sum_m \hat{u}_m(r) e^{im\theta}$



Harmonics $m = 1$ dominates in the region of interest.
Here $\alpha = 6.25 \times 10^{-6}$, but this holds for all the runs.

Fluctuations: turbulent energy profile

Behavior of harmonics explained by *zero modes of advection by the mean-flow*[16]

$$\begin{aligned}
\langle|\hat{u}_1|^2\rangle &= A_1 k_f^{-4/9}\delta^{-1/3}(\varepsilon L)^{2/3} + \cdots = A_1(\varepsilon R_u)^{2/3} + \cdots \ , \\
\langle|\hat{v}_1|^2\rangle &= (\varepsilon L)^{2/3}[A_1 k_f^{-4/9}\delta^{-1/3} + A_2 k_f^{-4/3}\delta^{-1}(\ell_f/r)^2] + \cdots \ , \\
\mathrm{Im}\langle\hat{u}_1\hat{v}_1^*\rangle &= A_1 k_f^{-4/9}\delta^{-1/3}(\varepsilon L)^{2/3} + \cdots \ ,
\end{aligned}$$

with $A_1, A_2 = O(1)$.

[16] A. Frishman and C. Herbert (submitted). *Phys. Rev. Lett.*

Fluctuations: turbulent energy profile

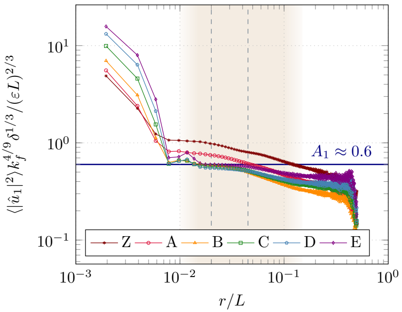
Behavior of harmonics explained by *zero modes of advection by the mean-flow*[16]

$$\langle|\hat{u}_1|^2\rangle = A_1 k_f^{-4/9}\delta^{-1/3}(\varepsilon L)^{2/3} + \cdots = A_1(\varepsilon R_u)^{2/3} + \cdots,$$

$$\langle|\hat{v}_1|^2\rangle = (\varepsilon L)^{2/3}[A_1 k_f^{-4/9}\delta^{-1/3} + A_2 k_f^{-4/3}\delta^{-1}(\ell_f/r)^2] + \cdots,$$

$$\mathrm{Im}\langle\hat{u}_1\hat{v}_1^*\rangle = A_1 k_f^{-4/9}\delta^{-1/3}(\varepsilon L)^{2/3} + \cdots,$$

with $A_1, A_2 = O(1)$.



[16] A. Frishman and C. Herbert (submitted). *Phys. Rev. Lett.*

Fluctuations: turbulent energy profile

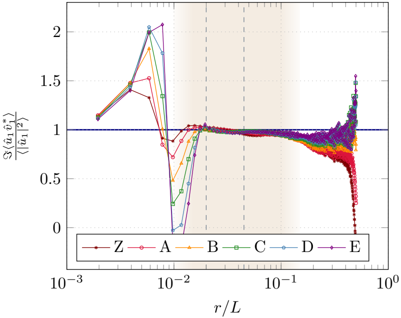
Behavior of harmonics explained by *zero modes of advection by the mean-flow*[16]

$$\langle|\hat{u}_1|^2\rangle = A_1 k_f^{-4/9}\delta^{-1/3}(\varepsilon L)^{2/3} + \cdots = A_1(\varepsilon R_u)^{2/3} + \cdots\,,$$

$$\langle|\hat{v}_1|^2\rangle = (\varepsilon L)^{2/3}[A_1 k_f^{-4/9}\delta^{-1/3} + A_2 k_f^{-4/3}\delta^{-1}(\ell_f/r)^2] + \cdots\,,$$

$$\mathrm{Im}\langle\hat{u}_1\hat{v}_1^*\rangle = A_1 k_f^{-4/9}\delta^{-1/3}(\varepsilon L)^{2/3} + \cdots\,,$$

with $A_1, A_2 = O(1)$.



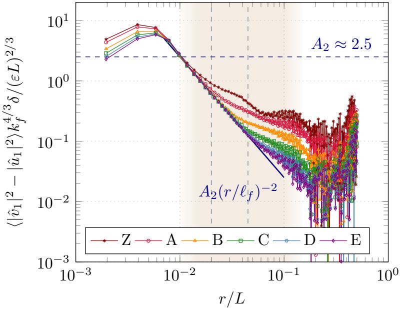
[16] A. Frishman and C. Herbert (submitted). *Phys. Rev. Lett.*

Fluctuations: turbulent energy profile

Behavior of harmonics explained by *zero modes of advection by the mean-flow*[16]

$$
\begin{aligned}
\langle |\hat{u}_1|^2 \rangle &= A_1 k_f^{-4/9} \delta^{-1/3} (\varepsilon L)^{2/3} + \cdots = A_1 (\varepsilon R_u)^{2/3} + \cdots , \\
\langle |\hat{v}_1|^2 \rangle &= (\varepsilon L)^{2/3} [A_1 k_f^{-4/9} \delta^{-1/3} + A_2 k_f^{-4/3} \delta^{-1} (\ell_f / r)^2] + \cdots , \\
\mathrm{Im} \langle \hat{u}_1 \hat{v}_1^* \rangle &= A_1 k_f^{-4/9} \delta^{-1/3} (\varepsilon L)^{2/3} + \cdots ,
\end{aligned}
$$

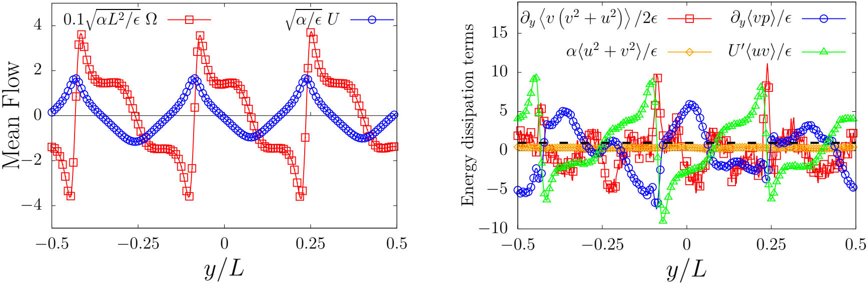
with $A_1, A_2 = O(1)$.



[16] A. Frishman and C. Herbert (submitted). *Phys. Rev. Lett.*

Prospect: Zonal jet profile and fluctuations

Is the jet profile determined by the forcing-advection balance $U'\langle uv\rangle = \epsilon$?[17]



Simulations by Jason Laurie (Aston University, UK)

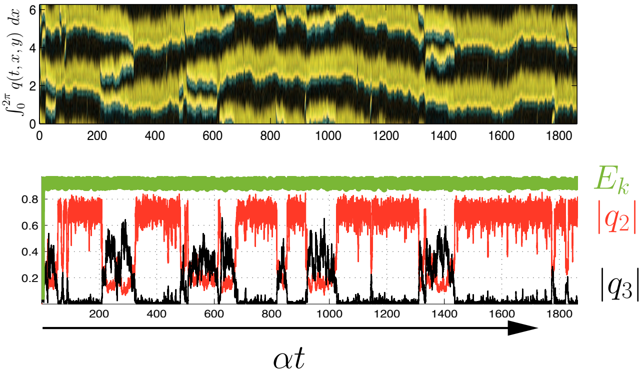
[17] E. Woillez and F. Bouchet (2017). *EPL*.

Prospect: Bistability in zonal jets

Zonal jets in the stochastic barotropic vorticity equation:

$$\partial_t \omega + \mathbf{u} \cdot \nabla \omega + \beta v = -\alpha \omega + \nu \Delta \omega + f_\omega$$

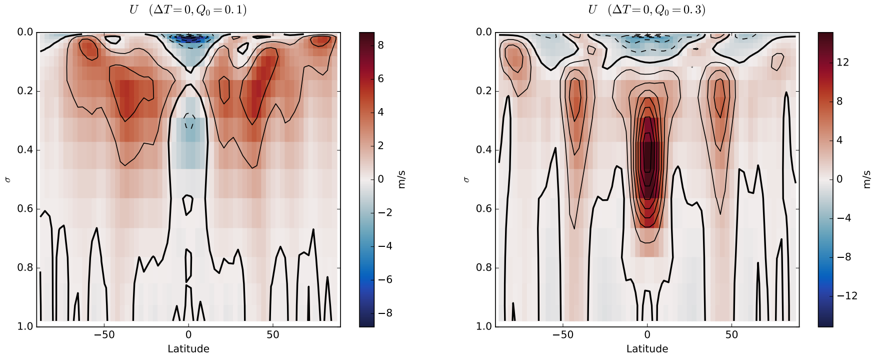
$$\beta = \mathbf{0.555}\beta_{\mathrm{mid}} \quad \alpha = 1.5 \cdot 10^{-3}$$



Simulations by Eric Simonnet (Inphyni, Nice).

Prospect: Bistability in zonal jets

Zonal wind in idealized GCM (T. Schneider, Caltech):



- Held-Suarez forcing ($T_{eq} = 315$ K):

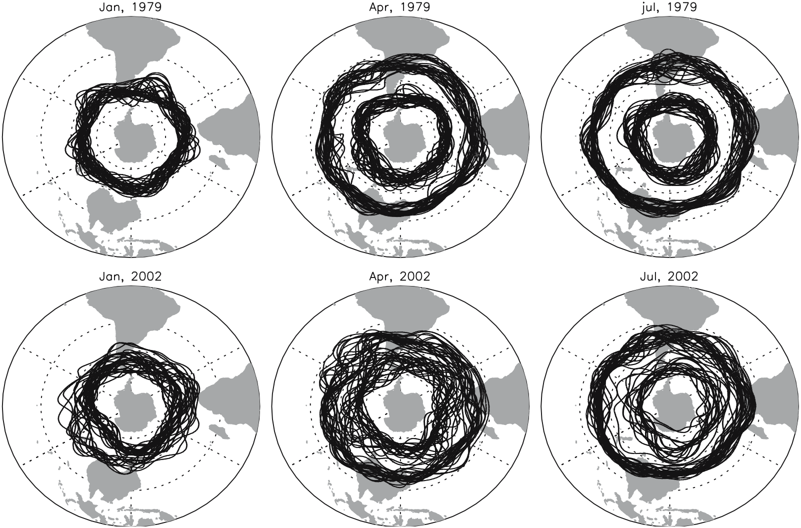
$$T_\star(p,\phi) = [T_{eq} - \Delta T \sin^2\phi - (\Delta\theta)_z \ln(p/p_0)\cos^2\phi](p/p_0)^{R/c_p}.$$

- Equatorial forcing:

$$Q(p,\phi,\lambda) = Q_0 \cos(k\lambda)\exp\left(-\frac{\phi^2}{\Delta\phi^2}\right)\sin\left(\pi\frac{p-p_t}{p_b-p_t}\right).$$

Prospect: Bistability in zonal jets

Jet Stream in the Southern Hemisphere[18]



[18] D. Gallego et al. (2005). *Clim. Dyn.*

Summary

Kinetic theory

- Asymptotic statistical closure of the Navier-Stokes equations based on timescale separation
- Reproduces emergence of large-scale coherent structures
- Allows explicit computations of mean-flow and fluctuations profiles in idealized context, in agreement with DNS

Prospects

- Reduced model to study slow dynamics of eddy-driven jets: attractors, fluctuations,...
- Combined with large deviation theory, study abrupt transitions in jet dynamics

Bouchet, F. et al. (2013). *J. Stat. Phys.* 153.4, pp. 572–625.
Cérou, F. and A. Guyader (2007). *Stoch. Anal. Appl.* 25, pp. 417–443.
Chen, Q. et al. (2003). *Phys. Rev. Lett.* 90, p. 214503.
Constantinou, N. C. et al. (2014). *J. Atmos. Sci.* 71.5, pp. 1818–1842.
Danilov, S. and D. Gurarie (2004). *Phys. Fluids* 16.7, p. 2592.
Farrell, B. F. and P. J. Ioannou (2003). *J. Atmos. Sci.* 60.17, pp. 2101–2118.
Farrell, B. F. and P. J. Ioannou (2007). *J. Atmos. Sci.* 64.10, pp. 3652–3665.
Frishman, A. and C. Herbert (submitted). *Phys. Rev. Lett.*
Gallego, D. et al. (2005). *Clim. Dyn.* 24.6, pp. 607–621.
Galperin, B and S Sukoriansky (2001). *Phys. Fluids* 13, p. 1545.
Gardiner, C. W. (2009). *Handbook of Stochastic Methods for physics, chemistry, and the natural sciences*. 4th edition. Springer, Berlin.
Giardina, C et al. (2011). *J. Stat. Phys.* 145, pp. 787–811.
Herbert, C. (2013). *J. Stat. Phys.* 152, pp. 1084–1114.
Herbert, C. and F. Bouchet (2017). *Phys. Rev. E* 96, 030201(R).
Kraichnan, R. H. (1967). *Phys. Fluids* 10, pp. 1417–1423.
Kramers, H. A. (1940). *Physica* 7, pp. 284–304.
Laurie, J. et al. (2014). *Phys. Rev. Lett.* 113, p. 254503.
Maltrud, M. E. and G. K. Vallis (1991). *J. Fluid Mech.* 228, pp. 321–342.
Marston, J. B. et al. (2016). *Phys. Rev. Lett.* 116, p. 214501.
O'Gorman, P. A. and T. Schneider (2007). *Geophys. Res. Lett.* 34, p. L22801.
Onsager, L. (1949). *Il Nuovo Cimento* 6, pp. 279–287.
Porco, C. C. et al. (2003). *Science* 299.5612, pp. 1541–1547.
Qi, W. and J. B. Marston (2014). *J. Stat. Mech.* P07020.
Qiu, B and S. M. Chen (2005). *J. Phys. Oceanogr.* 35.11, pp. 2090–2103.
Rhines, P. B. (1975). *J. Fluid Mech.* 69, pp. 417–443.
Robert, R. and J. Sommeria (1991). *J. Fluid Mech.* 229, pp. 291–310.
Schneider, T. and C. C. Walker (2006). *J. Atmos. Sci.* 63, pp. 1569–1586.
Smith, L. M. and V. Yakhot (1994). *J. Fluid Mech.* 274, p. 115.
Sommeria, J. (1986). *J. Fluid Mech.* 170, pp. 139–168.
Srinivasan, K. and W. R. Young (2012). *J. Atmos. Sci.* 69.5, pp. 1633–1656.
Tobias, S. M. and J. B. Marston (2013). *Phys. Rev. Lett.* 110, p. 104502.
Weeks, E. R. et al. (1997). *Science* 278, p. 1598.
Woillez, E. and F. Bouchet (2017). *EPL* 118, p. 54002. arXiv: `1609.00603v2 [physics.flu-dyn]`.
Xia, H et al. (2009). *Phys. Fluids* 21.12, p. 125101.
Zagarola, M. V. and A. J. Smits (1997). *Phys. Rev. Lett.* 78.2, pp. 239–242.

Can the mean-flow be described by an equilibrium state?

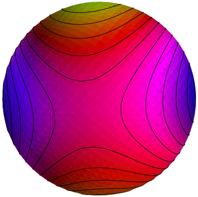
2D Euler equations: $\partial_t \omega + \mathbf{u} \cdot \nabla \omega = 0$.

Unlike 3D HIT, they have non-trivial equilibrium states[19].

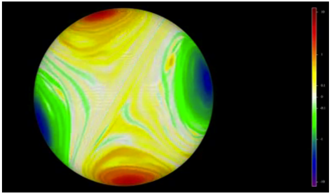
Invariants: $\int s(\omega(\mathbf{r})) d\mathbf{r}$.

At equilibrium, all the energy is in the mean-flow; no fluctuations.

Ex: flow on a sphere



Theoretical Equilibrium: Quadrupole[20]



DNS Final State[21]

[19] L. Onsager (1949). *Il Nuovo Cimento*; R. H. Kraichnan (1967). *Phys. Fluids*; R. Robert and J. Sommeria (1991). *J. Fluid Mech.*

[20] C. Herbert (2013). *J. Stat. Phys.*

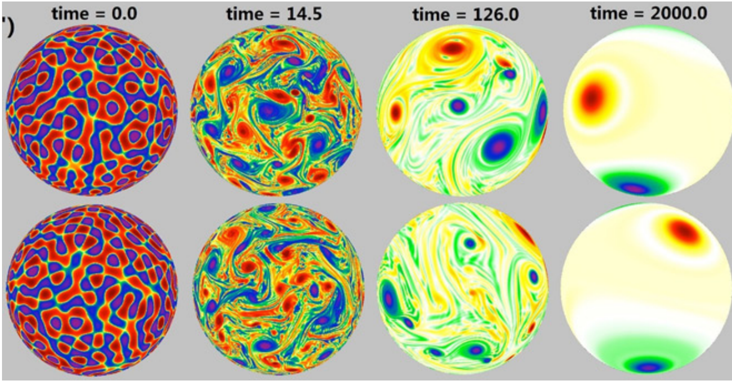
[21] W. Qi and J. B. Marston (2014). *J. Stat. Mech.*

The effect of rotation (DNS results[22])

When the Rossby waves are sufficiently slow, the system relaxes towards its equilibrium state.

“North Pole”

“South Pole”



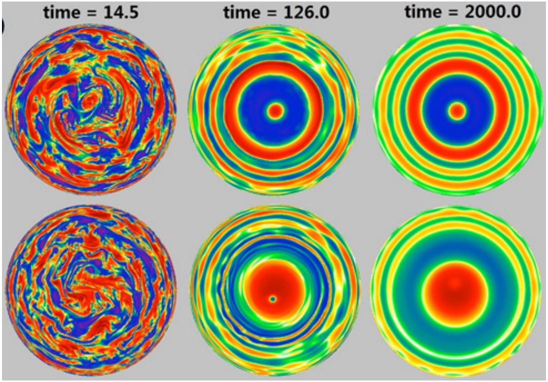
22 W. Qi and J. B. Marston (2014). *J. Stat. Mech.*

The effect of rotation (DNS results[22])

For faster rotation rates, Rossby waves arrest the cascade at the Rhines scale and lead to the emergence of zonal flows.

“North Pole”

“South Pole”



[22] W. Qi and J. B. Marston (2014). *J. Stat. Mech.*

The closure problem

Reynolds decomposition:

$$u_i = \bar{u}_i + u_i',$$

where $\bar{\cdot}$ is a projection operator.
The Navier-Stokes equations become:

$$\partial_t \bar{u}_i + \bar{u}_j \partial^j \bar{u}_i = -\partial_i \bar{P} + \nu \partial_j \partial^j \bar{u}_i - \partial^j \overline{u_i' u_j'},$$
$$\partial_t u_i' + \bar{u}_j \partial^j u_i' + u_j' \partial^j \bar{u}_i = -\partial_i P' + \nu \partial_j \partial^j u_i' - \partial^j u_i' u_j' + \partial^j \overline{u_i' u_j'}.$$

The major difficulty is to compute the Reynolds stress tensor $-\partial^j \overline{u_i' u_j'}$.

Modeling approaches:

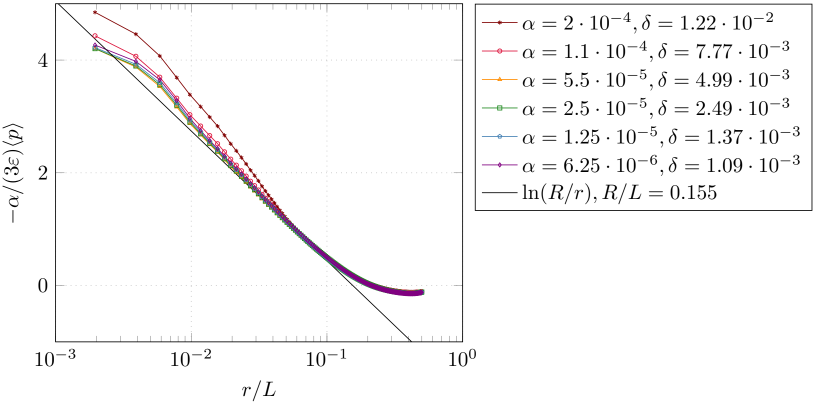
- *Large Eddy Simulations*: spatial filtering

$$\bar{u}_i(\mathbf{x}, t) = \int G(\mathbf{x} - \mathbf{y}) u_i(\mathbf{y}, t) d\mathbf{y}$$

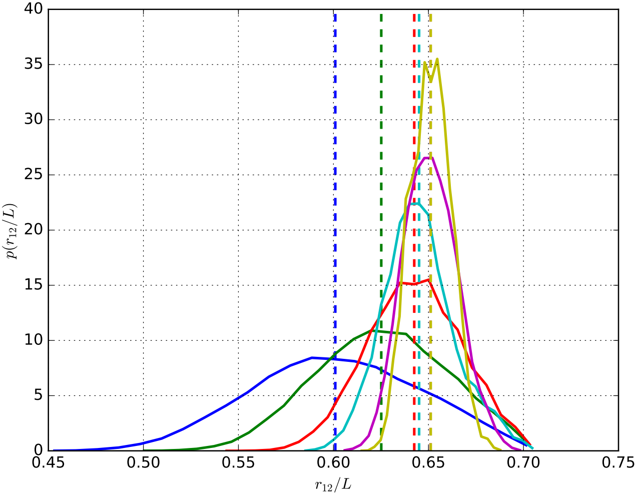
- *Reynolds Average Navier-Stokes*: time filtering

Mean pressure profile (DNS)

DNS: 512^2, $k_F = 100$, hyperviscosity, $\sim$ 300000 turnover times.



Intervortex distance

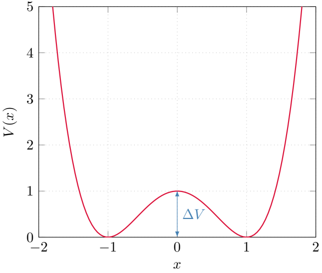


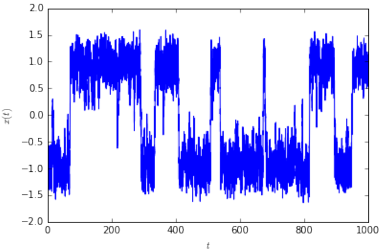
Simple stochastic process reproducing the dynamics of the inter-vortex distance?

Theoretical framework for noise induced transitions: the Kramers problem[23]

Overdamped Langevin dynamics:

$$\dot{x} = -V'(x) + \sqrt{2\epsilon}\eta, \quad V(x) = (x^2 - 1)^2, \quad \mathbb{E}[\eta(t)\eta(t')] = \delta(t - t').$$



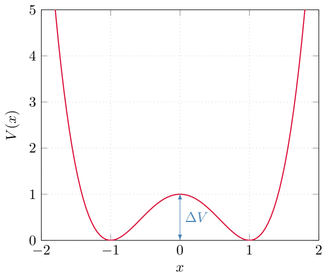


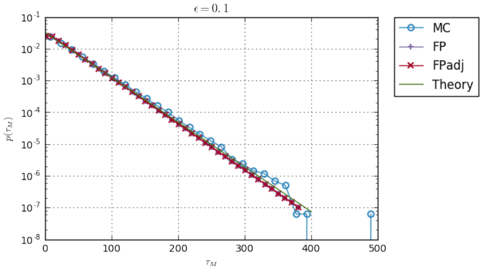
[23] H. A. Kramers (1940). *Physica*.

Theoretical framework for noise induced transitions: the Kramers problem[23]

Overdamped Langevin dynamics:

$$\dot{x} = -V'(x) + \sqrt{2\epsilon}\eta, \quad V(x) = (x^2 - 1)^2, \quad \mathbb{E}[\eta(t)\eta(t')] = \delta(t - t').$$





Transition probability

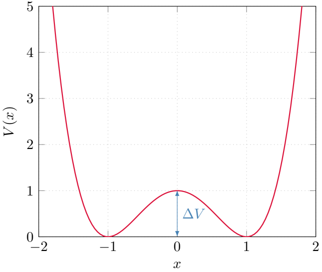
In the weak noise limit ($\epsilon \to 0$), transition times form a Poisson point process with transition rate $\lambda = \tau^{-1} e^{-\Delta V/\epsilon}$.
This is a large deviation result.

[23] H. A. Kramers (1940). *Physica*.

Theoretical framework for noise induced transitions: the Kramers problem[23]

Overdamped Langevin dynamics:

$$\dot{x} = -V'(x) + \sqrt{2\epsilon}\eta, \quad V(x) = (x^2-1)^2, \quad \mathbb{E}[\eta(t)\eta(t')] = \delta(t-t').$$



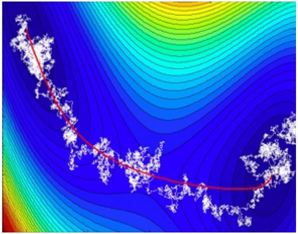


Fig. E. Vanden-Eijnden (Courant)

Instantons

Path integral formalism

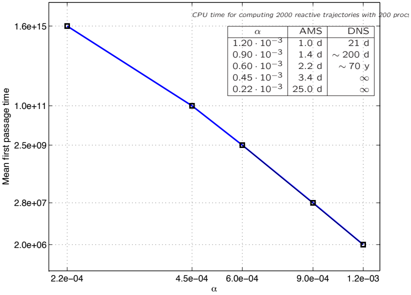
$$\mathbb{E}[\mathcal{O}] = \int \mathcal{D}[x]\mathcal{O}[x]\exp(-\mathcal{A}[x]/\epsilon), \quad \text{Action: } \mathcal{A}[x] = \frac{1}{4}\int dt(\dot{x} + V'(x))^2.$$

Instanton: most probable path: $\min_x\{\mathcal{A}[x] | x(-T) = -1, x(T) = 1\}$.

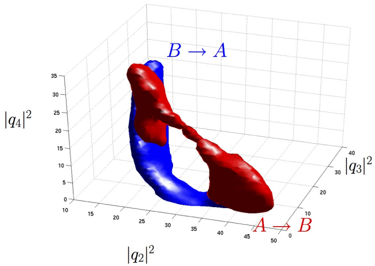
[23] H. A. Kramers (1940). *Physica*.

Arrhenius law and Instantons in jet transitions

Numerical algorithms to compute large deviations: dynamics biased in a controlled way[24].



Arrhenius law



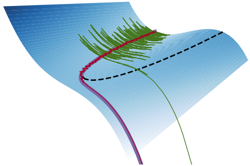
"Instantons"

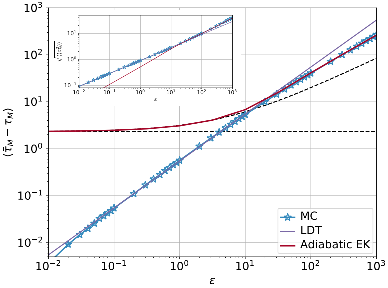
Jet transition simulations with rare event algorithm (AMS) by Eric Simonnet (Inphyni).

[24] C Giardina et al. (2011). *J. Stat. Phys.* F. Cérou and A. Guyader (2007). *Stoch. Anal. Appl.*

Escape in stochastic saddle-node bifurcation[25]

$$dx_t = (x_t^2 + t)dt + \sqrt{2\epsilon}dW_t, \qquad \tau_M = \inf\{t \geq t_0, x_t \geq M\}$$





Competition between deterministic and stochastic effects.

[25] C. Herbert and F. Bouchet (2017). *Phys. Rev. E*.