Modular curves of prime-power level with infinitely many rational points

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Galois representations

Let E be an elliptic curve over $\mathbb{Q}$ and let $N \geq 1$ be an integer.

The Galois group $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ acts on the N-torsion subgroup of $E(\overline{\mathbb{Q}})$

$$E[N] \simeq \mathbb{Z}/N\mathbb{Z} \oplus \mathbb{Z}/N\mathbb{Z},$$

via its coordinate-wise action on points. This yields a representation

$$\rho_{E,N}\colon \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{Aut}(E[N]) \simeq \mathrm{GL}_2(\mathbb{Z}/N\mathbb{Z}),$$

whose image we denote $G_E(N)$. By choosing bases compatibly, we can take inverse limits to obtain the adelic representation

$$\rho_E\colon \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \varprojlim_N \mathrm{GL}_2(\mathbb{Z}/N\mathbb{Z}) \simeq \mathrm{GL}_2(\hat{\mathbb{Z}}) \simeq \prod_\ell \mathrm{GL}_2(\mathbb{Z}_\ell).$$

whose image G_E is equipped with projection maps $\pi_N\colon G_E \to G_E(N)$.

Modular curves

Let $F_N := \mathbb{Q}(\zeta_N)(X(N))$. Then $F_1 = \mathbb{Q}(j)$ and F_N/F_1 is Galois with

$$\mathrm{Gal}(F_N/F_1) \simeq \mathrm{GL}_2(\mathbb{Z}/N\mathbb{Z})/\{\pm I\}.$$

Let $G \subseteq \mathrm{GL}_2(\mathbb{Z}/N\mathbb{Z})$ be a group containing $-I$ with $\det(G) = (\mathbb{Z}/N\mathbb{Z})^\times$.

The Galois action of $\alpha \in G$ on $f(\tau) = \sum a_n q^{n/N} \in F_N$ is defined by:

- $\alpha f := f(\gamma^{\mathrm{t}}\tau)$ if $\alpha \equiv \gamma \bmod N$ with $\gamma \in \mathrm{SL}_2(\mathbb{Z})$;
- $\left(\begin{smallmatrix}1 & 0\\ 0 & d\end{smallmatrix}\right) f := \sum \sigma_d(a_n) q^{n/N}$ where $\sigma_d(\zeta_N) := \zeta_N^d$.

Definition: *$X_G/\mathbb{Q}$ is the smooth projective curve with function field F_N^G. The morphism $J_G\colon X_G \to X(1)$ is given by the inclusion $F_1 \subseteq F_N^G$.*

If $H = \pi_M(\pi_N^{-1}(G))$ and $\pi_N^{-1}(G) = \pi_M^{-1}(H)$, then $X_G = X_H$.
The least such M is the *level* of G and X_G; we identify G and $\pi_N^{-1}(G)$.

Proposition: *For $E/\mathbb{Q}$ with $j(E) \notin \{0, 1728\}$, up to conjugacy in* $\mathrm{GL}_2(\hat{\mathbb{Z}})$:

$$G_E \subseteq G \iff j(E) \in J_G(X_G(\mathbb{Q})).$$

Congruence subgroups

Given $G \subseteq \mathrm{GL}_2(\hat{\mathbb{Z}})$ of level N, let $\Gamma \subseteq \mathrm{SL}_2(\mathbb{Z})$ be the preimage of $\pi_N(G) \cap \mathrm{SL}_2(\mathbb{Z}/N\mathbb{Z})$ under the projection map $\mathrm{SL}_2(\mathbb{Z}) \twoheadrightarrow \mathrm{SL}_2(\mathbb{Z}/N\mathbb{Z})$.

Then Γ is a congruence subgroup of level N, and the modular curve $X_\Gamma := \Gamma\backslash\mathfrak{h}^*$ is isomorphic to the base change of X_G to $\mathbb{C}$.

The genus g of X_G and X_Γ must coincide, but their levels need not! (the level of X_Γ necessarily divides the level of X_G).

For each $g \geq 0$ we have $g(X_\Gamma) = g$ for only finitely many Γ [Den75]. For $g \leq 24$ these Γ can be found in the tables of Cummins and Pauli.

By contrast, we may have $g(X_G) = g$ for infinitely many X_G.

Call $g(X_G)$ the genus of G.

Restricting the level

For each odd prime p there is a $G \subseteq \mathrm{GL}_2(\mathbb{Z}/2p\mathbb{Z})$ of index 2 that surjects on to both $\mathrm{GL}_2(\mathbb{Z}/2\mathbb{Z})$ and $\mathrm{GL}_2(\mathbb{Z}/p\mathbb{Z})$.

The corresponding X_G are all genus 0 curves isomorphic to $\mathbb{P}^1_{\mathbb{Q}}$ with maps $J_G\colon X_G \to X(1)$ given by $J_G(t) = 1728 + \left(\frac{-1}{p}\right)pt^2$.

One can similarly construct infinite families of genus 0 curves $X_G \simeq \mathbb{P}^1_{\mathbb{Q}}$ of level N divisible by any of $3, 5, 7, 13$.

And one can take combinations: for example, there are 12 non-conjugate G of level 91 and index at least 24 with $X_G \simeq \mathbb{P}^1_{\mathbb{Q}}$.

So we really need to restrict the level in order to get a finite problem. We have the following previous results for prime level [Z15a, S15].

Theorem: *29 X_G of prime level have infinitely many rational points*.

Conjecture: *64 X_G of prime level have a non-cuspidal, non-CM rational point*.

Main result

Theorem: 248 modular curves X_G of prime power level infinitely many rational points. Of these, 220 have genus 0 and 28 have genus 1.[1]

For each of these 248 curves we give explicit maps $J_G \colon X_G \to X(1)$.

Example: There is a unique X_G of genus 0 and level 27 for which $X_G(\mathbb{Q})$ is infinite. It arises for the group

$$G := \left\langle \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right), \left(\begin{smallmatrix} 2 & 1 \\ 9 & 5 \end{smallmatrix}\right), \left(\begin{smallmatrix} 1 & 2 \\ 3 & 2 \end{smallmatrix}\right) \right\rangle \subseteq \mathrm{GL}_2(\mathbb{Z}/27\mathbb{Z}).$$

The morphism $J_G \colon X_G \to X(1)$ is given by

$$J_G(t) = \frac{(t^3+3)^3(t^9+9t^6+27t^3+3)^3}{t^3(t^6+9t^3+27)}.$$

Remark: These maps can be used to construct models of modular curves of higher (possibly composite) level as fiber products.

[1] The X_G of 2-power level were also found by Rouse and Zureick-Brown [RZB15].

Implications

For the groups G in our main theorem, we have an explicit description of the set of j-invariants of elliptic curves $E/\mathbb{Q}$ for which $G_E \subseteq G$.

This effectively determines the possible images $G_E(\ell^\infty)$ of all ℓ-adic Galois representations

$$\rho_{E,\ell^\infty}\colon\ \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to\ \mathrm{GL}_2(\mathbb{Z}_\ell)$$

for $E/\mathbb{Q}$, up to an (ineffective!) finite set of exceptional j-invariants.

Theorem: For $\ell = 2, 3, 5, 7, 11, 13$ there are $1201, 47, 23, 15, 2, 11$ subgroups H of $\mathrm{GL}_2(\mathbb{Z}_\ell)$ that arise as $G_E(\ell^\infty)$ for infinitely many $E/\mathbb{Q}$ with distinct j-invariants; for $\ell > 13$ there is only $H = \mathrm{GL}_2(\mathbb{Z}_\ell)$.

Corollary: For each prime ℓ there is a finite set $\mathcal{S}_\ell$ such that for $E/\mathbb{Q}$, either $j(E) \in \mathcal{S}_\ell$ or $G_E(\ell^\infty)$ is one of the groups H in the theorem.

Note that the exceptional sets S_ℓ all include the 13 CM j-invariants.

Finiteness results

Call an open subgroup $G \subseteq \mathrm{GL}_2(\hat{\mathbb{Z}})$ *admissible* if

(i) $\det(G) = \hat{\mathbb{Z}}^\times$;

(ii) G contains an element $\mathrm{GL}_2(\hat{\mathbb{Z}})$-conjugate to $\left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$ or $\left(\begin{smallmatrix} 1 & 1 \\ 0 & -1 \end{smallmatrix}\right)$ (in other words, G admits complex conjugation);

(iii) $-I \in G$ (in other words, G contains all its "twists").

For $E/\mathbb{Q}$ without CM, G_E automatically satisfies (i) and (ii), and $G_{E'}$ satisfies (iii) for all but finitely many quadratic twists E' of E.

Proposition: *For each $g \geq 0$, only finitely many admissible G of prime power level have genus g; these G can be effectively enumerated.*

Lemma: *Let $G \subseteq \mathrm{GL}_2(\hat{\mathbb{Z}})$ and define $i_n := [\mathrm{GL}_2(\mathbb{Z}/\ell^n\mathbb{Z}) : \pi_{\ell^n}(G)]$. If $i_{n+1} = i_n$ for any $n \geq 2$ $(n \geq 1$ if $\ell > 2)$ then $i_m = i_n$ for all $m \geq n$.*

Corollary: *Among the admissible groups G of prime power level, there are exactly 220 of genus 0 and 250 of genus 1.*

Admissible groups of genus 0 with $X_G(\mathbb{Q})$ infinite.

Theorem: Let G be admissible, of prime power level and genus 0. Then $X_G(\mathbb{Q})$ is infinite.

Proof: Let G be an admissible group of ℓ-power level and genus 0.

- Admissibility ensures $X_G(\mathbb{R}) \neq \emptyset$, since G admits complex conjugation. Indeed, for any $E/\mathbb{Q}$ we have $\rho_E(\overline{\mathbb{Q}}/(\overline{\mathbb{Q}} \cap \mathbb{R})) \subseteq G$.
- For $p \nmid \ell$ we must have $X_G(\mathbb{Q}_p) \neq \emptyset$, since $\overline{X}_G(\mathbb{F}_p) \neq \emptyset$ (by Hensel).
- Therefore $X_G(\mathbb{Q}_\ell) \neq \emptyset$ (by Hilbert).
- Thus $X_G(\mathbb{Q}_p) \neq \emptyset$ for all $p \leq \infty$, so $X_G(\mathbb{Q}) \neq \emptyset$ (by Hasse).

The genus 0 curve X_G has a rational point, so $X_G \simeq \mathbb{P}^1$. □

Remark: The pointless conics in [RZB15] all occur for G that are inadmissible because they do not admit complex conjugation.

Admissible groups of genus 1 with $X_G(\mathbb{Q})$ infinite.

If $g = 1$ then $J_G := \mathrm{Jac}(X_G)$ is an elliptic curve whose conductor is a power of a prime ℓ. There are only finitely many such elliptic curves, all of which can be found in Cremona's tables.

For a finite set S of small primes $p \nmid \ell$, their a_p-values distinguish their isogeny classes, and therefore their ranks.

For each of these $p \nmid \ell$ we compute $a_p = p + 1 - \#X_G(\mathbb{F}_p)$ by counting $\mathbb{F}_p$-points on X_G (as explained on the next slide).

If $J_G(\mathbb{Q})$ has rank 0 then $X_G(\mathbb{Q})$ is finite.
If $J_G(\mathbb{Q})$ has positive rank, either $X_G(\mathbb{Q}) = \emptyset$ or $X_G(\mathbb{Q})$ is infinite.
To distinguish which applies it suffices to find $E/\mathbb{Q}$ with $G_E \subseteq G$.

Result: We find that only 28 of the 250 admissible groups of prime power level and genus 1 correspond to E_G with positive rank.
For each of these we can exhibit $E/\mathbb{Q}$ with $G_E \subseteq G$.

Counting $\mathbb{F}_p$-points on X_G

$$\#X_G(\mathbb{F}_p) = \sum_{j\in\mathbb{F}_p} \#\{P \in Y_G(\mathbb{F}_p) : J_G(P) = j\} + \#X_G^\infty(\mathbb{F}_p)$$

To count the set of $\mathbb{F}_p$-rational cusps $X_G^\infty(\mathbb{F}_p)$ we count double cosets in $G\backslash \mathrm{GL}_2(\mathbb{Z}/N\mathbb{Z})/\langle \pm \left(\begin{smallmatrix}1&1\\0&1\end{smallmatrix}\right)\rangle$ fixed by the right action of $\chi_N(\mathrm{Frob}_p)$.

To compute the sum, for each $j \in \mathbb{F}_p$, fix any $\overline{E}/\mathbb{F}_p$ with $j(\overline{E}) = j$.[2]

By a theorem of Duke and Tóth [DT02], there is an integer matrix A_p with $\operatorname{tr} A_p = p + 1 - \#\overline{E}(\mathbb{F}_p)$ and $\det A_p = p$ such that, up to conjugacy,

$$\rho_{E,N}(\mathrm{Frob}_p) \equiv A_p \bmod N,$$

for any $E \equiv \overline{E} \bmod p$ and $N \perp p$; it can be efficiently computed [S15].

We can then compute $\#\{P \in Y_G(\mathbb{F}_p) : J_G(P) = j\}$ as the number of right cosets of $G \subseteq \mathrm{GL}_2(\mathbb{Z}/N\mathbb{Z})$ fixed by right multiplication by A_p (this amounts to computing the permutation character $\chi_G(A_p)$).

Key point: We don't need a model for X_G!

[2] For $j = 0, 1728$ one needs to modify this procedure.

Constructing the genus 0 maps $J_G\colon X_G \to X(1)$

We first search for distinct points j_0, j_1, j_∞ in the image of J_G by computing $G_{E,N}$ for many $E/\mathbb{Q}$ using a fast Monte Carlo algorithm.

There exists $J \in \mathbb{Q}(t)$ such that $J(0) = j_0$, $J(1) = j_1$, $J(\infty) = j_\infty$ for which $\mathbb{Q}(X_G) = \mathbb{Q}(f)$ where $f \in \mathbb{Q}(X_G)$ satisfies $J(f) = j$.

Let $\Gamma \subseteq \mathrm{SL}_2(\mathbb{Z})$ be the inverse image of $\pi_N(G) \cap \mathrm{SL}_2(\mathbb{Z}/N\mathbb{Z})$. Then $\mathbb{C}(X_\Gamma) = \mathbb{C}(h)$ for some modular function h of level N with a unique pole at the cusp ∞ (the function h is a *hauptmodul* for Γ). We compute h using Siegel functions (see paper for details).

We have $\mathbb{Q}(\zeta_N)(X_G) = \mathbb{Q}(\zeta_N)(h)$ and $j = J'(h)$ for some $J' \in \mathbb{Q}(\zeta_N)(t)$ we can compute. Then $J(t) = J'(\psi(t))$ for some degree-1 ψ such that $J'(\psi(0)) = j_0$, $J'(\psi(1)) = j_1$, $J'(\psi(\infty)) = j_\infty$ (finitely many cases).

Now let $f := \psi^{-1}(h)$ and check whether G fixes f (if not, wrong ψ). We then have $J(f) = J'(h) = j$ as desired.

Key point: We can unconditionally verify that $\mathbb{Q}(X_G) = \mathbb{Q}(f)$ at the end (if not, redo the Monte Carlo step). This yields a Las Vegas algorithm.

#	group	i	N	generators	map	sup
0	1A^0-1a	1	1			
1	3A^0-3a	3	3	[0, 1, 2, 0], [1, 1, 1, 2], [1, 0, 0, 2]	t^3	0
2	3B^0-3a	4	3	[0, 1, 2, 1], [1, 2, 0, 2]	$(t+3)^3(t+27)\ /\ t$	0
3	3C^0-3a	6	3	[0, 1, 2, 0], [1, 0, 0, 2]	$(t-9)(t+3)\ /\ t$	1
4	3D^0-3a	12	3	[2, 0, 0, 2], [1, 0, 0, 2]	$729\ /\ (t^3-27)$	2
					$-27(t-3)\ /\ (t^2+3t+9)$	3
5	9A^0-9a	9	9	[0, 2, 4, 0], [1, 1, 4, 5], [1, 0, 0, 2]	t^3+9t-6	1
6	9B^0-9a	12	9	[1, 1, 0, 1], [2, 0, 0, 5], [1, 0, 0, 2]	$t(t^2+9t+27)$	2
7	9C^0-9a	12	9	[2, 0, 0, 5], [4, 2, 3, 4], [1, 0, 0, 2]	t^3	2
8	9D^0-9a	18	9	[2, 0, 0, 5], [1, 3, 3, 1], [0, 2, 4, 0], [1, 0, 0, 2]	$-27\ /\ t^3$	3
					$(t^2-3)\ /\ t$	5
9	9E^0-9a	18	9	[1, 3, 0, 1], [2, 1, 1, 1], [4, 2, 0, 5]	$-9(t^3+3t^2-9t-3)/(8t^3)$	1
10	9F^0-9a	27	9	[0, 2, 4, 1], [4, 3, 5, 4], [4, 5, 0, 5]	$\frac{3^7(t^2-1)^3(t^6+3t^5+6t^4+t^3-3t^2+12t+16)^3}{(2t^3+3t^2-3t-5)^{-1}(t^3-3t-1)^9}$	0
11	9G^0-9a	27	9	[0, 4, 2, 3], [5, 1, 1, 4], [5, 3, 0, 4]	$\frac{(t^3-9t-12)(9-3t^3)(5t^3+18t^2+18t+3)}{(t^3+3t^2-3)^3}$	1
12	9H^0-9a	36	9	[1, 3, 0, 1], [5, 0, 3, 2], [1, 0, 2, 2]	$3(t^3+9)/t^3$	4
					$3t\ /\ (2t^2-3t+6)$	9
13	9H^0-9b	36	9	[1, 3, 0, 1], [5, 0, 3, 2], [2, 1, 0, 1]	$3(t^3+9t^2-9t-9)\ /\ (t^3-9t^2-9t+9)$	4
14	9H^0-9c	36	9	[1, 3, 0, 1], [5, 0, 3, 2], [4, 2, 0, 5]	$-6(t^3-9t)\ /\ (t^3+9t^2-9t-9)$	4
					$-(t^2+3)\ /\ (t^2+8t+3)$	10
15	9I^0-9a	36	9	[2, 1, 0, 5], [1, 2, 3, 2]	$-6(t^3-9t)\ /\ (t^3-3t^2-9t+3)$	6
16	9I^0-9b	36	9	[2, 1, 0, 5], [4, 0, 3, 5]	$-3(t^3+9t^2-9t-9)\ /\ (t^3+3t^2-9t-3)$	6
17	9I^0-9c	36	9	[2, 2, 0, 5], [2, 2, 3, 1]	$(t^3-6t^2+3t+1)\ /\ (t^2-t)$	6
18	9J^0-9a	36	9	[1, 3, 0, 1], [2, 2, 3, 8], [1, 2, 0, 2]	$(t^3-3t+1)\ /\ (t^2-t)$	7
19	9J^0-9b	36	9	[1, 3, 0, 1], [2, 2, 3, 8], [2, 1, 0, 1]	$-18(t^2-1)\ /\ (t^3-3t^2-9t+3)$	7
20	9J^0-9c	36	9	[1, 3, 0, 1], [5, 2, 3, 5], [4, 0, 0, 5]	$3(t^3+3t^2-9t-3)\ /\ (t^3-3t^2-9t+3)$	7
21	27A^0-27a	36	27	[1, 1, 0, 1], [2, 1, 9, 5], [1, 2, 3, 2]	t^3	6

group	i	N	generators	map	supergroup
$1A^0$-1a	1	1			
$5A^0$-5a	5	5	[2, 1, 0, 3], [1, 2, 2, 0], [1, 1, 0, 2]	$t^3(t^2+5t+40)$	$1A^0$-1a
$5B^0$-5a	6	5	[2, 0, 0, 3], [1, 0, 1, 1], [1, 0, 0, 2]	$(t^2+10t+5)^3 / t$	$1A^0$-1a
$5C^0$-5a	10	5	[3, 1, 0, 2], [1, 2, 2, 0], [2, 2, 2, 1]	$8000t^3(t+1)(t^2-5t+10)^3 / (t^2-5)^5$	$1A^0$-1a
$5D^0$-5a	12	5	[4, 0, 1, 4], [1, 0, 0, 2]	$125t / (t^2-11t-1)$	$5B^0$-5a
$5D^0$-5b	12	5	[4, 0, 1, 4], [2, 0, 0, 1]	$(t^2-11t-1) / t$	$5B^0$-5a
$5E^0$-5a	15	5	[2, 1, 0, 3], [2, 0, 2, 3], [1, 0, 2, 2]	$(t+5)(t^2-5) / (t^2+5t+5)$	$5A^0$-5a
$5G^0$-5a	30	5	[3, 1, 0, 2], [2, 1, 0, 1]	$125 / (t(t^4+5t^3+15t^2+25t+25))$	$5E^0$-5a
				$(t^2+5)/t$	$5B^0$-5a
$5G^0$-5b	30	5	[3, 1, 0, 2], [2, 1, 3, 3]	$-t(t^2+5t+10) / (t^3+5t^2+10t+10)$	$5C^0$-5a
				$-5(t^2+4t+5)/(t^2+5t+5)$	$5E^0$-5a
$5H^0$-5a	60	5	[4, 0, 0, 4], [2, 0, 0, 1]	$-1 / t^5$	$5D^0$-5a
				$\frac{-(t^4-2t^3+4t^2-3t+1)}{t(t^4+3t^3+4t^2+2t+1)}$	$5D^0$-5b
				$5t / (t^2-t-1)$	$5G^0$-5a
$25A^0$-25a	30	25	[2, 2, 0, 13], [4, 1, 3, 1], [2, 3, 0, 6]	$(t-1)(t^4+t^3+6t^2+6t+11)$	$5B^0$-5a
$25B^0$-25a	60	25	[9, 10, 0, 14], [0, 7, 7, 2], [2, 8, 0, 1]	$-t^5$	$5D^0$-5b
				$(1-t^2) / t$	$25A^0$-25a
$25B^0$-25b	60	25	[9, 10, 0, 14], [0, 7, 7, 2], [4, 1, 0, 7]	$\frac{-(t^4-2t^3+4t^2-3t+1)}{t(t^4+3t^3+4t^2+2t+1)}$	$5D^0$-5a
				$(t^2+4t-1) / (t^2-t-1)$	$25A^0$-25a
$7B^0$-7a	8	7	[2, 0, 0, 4], [3, 0, 1, 5], [1, 0, 0, 3]	$(t^2+5t+1)^3(t^2+13t+49) / t$	$1A^0$-1a
$7D^0$-7a	21	7	[0, 3, 2, 3], [2, 4, 4, 5], [3, 1, 0, 4]	$\frac{(2t-1)^3(t^2-t+2)^3(2t^2+5t+4)^3(5t^2+2t-4)^3}{(t^3+2t^2-t-1)^7}$	$1A^0$-1a
$7E^0$-7a	24	7	[6, 0, 1, 6], [1, 0, 0, 3]	$49(t^2-t) / (t^3-8t^2+5t+1)$	$7B^0$-7a
$7E^0$-7b	24	7	[6, 0, 1, 6], [3, 0, 0, 1]	$(t^3-8t^2+5t+1) / (t^2-t)$	$7B^0$-7a
$7E^0$-7c	24	7	[6, 0, 1, 6], [3, 0, 0, 4]	$-7(t^3-2t^2-t+1) / (t^3-t^2-2t+1)$	$7B^0$-7a
$7F^0$-7a	28	7	[3, 1, 4, 4], [4, 4, 1, 3], [3, 4, 0, 4]	$\frac{t(t+1)^3(t^2-5t+1)^3(t^2-5t+8)^3}{(t^4-5t^3+8t^2-7t+7)^{-3}(t^3-4t^2+3t+1)^7}$	$1A^0$-1a
$13A^0$-13a	14	13	[2, 0, 0, 7], [1, 0, 1, 1], [1, 0, 0, 2]	$(t^2+5t+13)(t^4+7t^3+20t^2+19t+1)^3 / t$	$1A^0$-1a
$13B^0$-13a	28	13	[3, 0, 0, 9], [4, 0, 1, 10], [1, 0, 0, 2]	$13t / (t^2-3t-1)$	$13A^0$-13a
$13B^0$-13b	28	13	[3, 0, 0, 9], [4, 0, 1, 10], [2, 0, 0, 1]	$(t^2-3t-1) / t$	$13A^0$-13a
$13C^0$-13a	42	13	[5, 0, 0, 8], [1, 0, 1, 1], [1, 0, 0, 2]	$13(t^2-t) / (t^3-4t^2+t+1)$	$13A^0$-13a
$13C^0$-13b	42	13	[5, 0, 0, 8], [1, 0, 1, 1], [2, 0, 0, 1]	$(t^3-4t^2+t+1) / (t^2-t)$	$13A^0$-13a
$13C^0$-13c	42	13	[5, 0, 0, 8], [1, 0, 1, 1], [2, 0, 0, 3]	$-(5t^3-7t^2-8t+5) / (t^3-4t^2+t+1)$	$13A^0$-13a

group	i	N	generators	map	supergroup
$1A^0$-1a	1	1			
$2A^0$-2a	2	2	[0, 1, 1, 1]	$t^2 + 1728$	$1A^0$-1a
$2A^0$-4a	2	4	[1, 2, 0, 1], [1, 1, 1, 2], [1, 1, 0, 3]	$-t^2 + 1728$	$1A^0$-1a
$2A^0$-8a	2	8	[1, 2, 0, 1], [1, 1, 1, 2], [1, 0, 0, 3], [1, 1, 0, 5]	$-2t^2 + 1728$	$1A^0$-1a
$2A^0$-8b	2	8	[1, 2, 0, 1], [1, 1, 1, 2], [1, 1, 0, 3], [1, 1, 0, 5]	$2t^2 + 1728$	$1A^0$-1a
$2B^0$-2a	3	2	[0, 1, 1, 0]	$(256 - t)^3/t^2$	$1A^0$-1a
$2C^0$-2a	6	2		$-t^2 + 64$	$2B^0$-2a
$2C^0$-4a	6	4	[1, 2, 0, 1], [3, 0, 0, 3], [1, 0, 2, 1], [1, 1, 0, 3]	$64(t^2 + 1)$	$2B^0$-2a
$2C^0$-8a	6	8	[1, 2, 0, 1], [3, 0, 0, 3], [1, 0, 2, 1], [1, 0, 0, 3], [0, 1, 1, 0]	$32(t^2 + 2)$	$2B^0$-2a
$2C^0$-8b	6	8	[1, 2, 0, 1], [3, 0, 0, 3], [1, 0, 2, 1], [1, 1, 0, 3], [1, 1, 0, 5]	$-32(t^2 - 2)$	$2B^0$-2a
$4A^0$-4a	4	4	[1, 1, 1, 2], [0, 1, 3, 0], [1, 1, 0, 3]	$4t^3(8 - t)$	$1A^0$-1a
$4B^0$-4a	6	4	[3, 0, 0, 3], [0, 1, 3, 2], [1, 0, 0, 3]	$256/(t^2 + 4)$	$2B^0$-2a
$4B^0$-4b	6	4	[3, 0, 0, 3], [0, 1, 3, 2], [1, 2, 0, 3]	$-4096/(t^2 + 16t)$	$2B^0$-2a
$4B^0$-8a	6	8	[3, 0, 0, 3], [1, 4, 0, 1], [0, 3, 5, 2], [1, 0, 0, 3], [1, 2, 0, 5]	$-512/(t^2 - 8)$	$2B^0$-2a
$4B^0$-8b	6	8	[3, 0, 0, 3], [1, 4, 0, 1], [0, 3, 5, 2], [1, 2, 0, 3], [1, 2, 0, 5]	$512/(t^2 + 8)$	$2B^0$-2a
$4C^0$-4a	6	4	[1, 2, 2, 1], [0, 1, 3, 0], [1, 0, 0, 3]	t^2	$2B^0$-2a
$4C^0$-4b	6	4	[1, 2, 2, 1], [0, 1, 3, 0], [1, 2, 0, 3]	$-t^2$	$2B^0$-2a
$4C^0$-8a	6	8	[1, 4, 0, 1], [2, 1, 3, 2], [0, 3, 5, 0], [1, 0, 0, 3], [1, 2, 0, 5]	$-128t^2$	$2B^0$-2a
$4C^0$-8b	6	8	[1, 4, 0, 1], [2, 1, 3, 2], [0, 3, 5, 0], [1, 2, 0, 3], [1, 2, 0, 5]	$128t^2$	$2B^0$-2a
$4D^0$-4a	8	4	[2, 1, 1, 3], [1, 1, 0, 3]	$-(t^2 + 2t - 2)/t$	$4A^0$-4a
$4D^0$-8a	8	8	[1, 4, 0, 1], [2, 1, 5, 3], [3, 3, 0, 5], [0, 1, 3, 0]	$(t^2 + 4t - 2)/(4 - 2t^2)$	$4A^0$-4a
$4E^0$-4a	12	4	[3, 0, 0, 3], [1, 2, 2, 1], [0, 1, 1, 0]	$(t^2 - 1)/(2t)$	$2C^0$-4a
$4E^0$-4b	12	4	[3, 0, 0, 3], [1, 2, 2, 1], [1, 0, 0, 3]	$8(2t^2 + 1)$	$2C^0$-2a
$4E^0$-4c	12	4	[3, 0, 0, 3], [1, 2, 2, 1], [1, 2, 0, 3]	$4(t^2 + 1)/t$	$2C^0$-2a
$4E^0$-8a	12	8	[3, 0, 0, 3], [1, 4, 0, 1], [5, 0, 0, 5], [1, 2, 2, 5], [1, 0, 0, 3], [0, 1, 1, 0]	$4t/(t^2 - 2)$	$2C^0$-8a
$4E^0$-8b	12	8	[3, 0, 0, 3], [1, 4, 0, 1], [5, 0, 0, 5], [1, 2, 2, 5], [1, 0, 0, 3], [1, 2, 0, 5]	$8(t^2 - 1)$	$2C^0$-2a
$4E^0$-8c	12	8	[3, 0, 0, 3], [1, 4, 0, 1], [5, 0, 0, 5], [1, 2, 2, 5], [1, 2, 0, 3], [0, 1, 1, 0]	$(t^2 - 2)/(2t)$	$2C^0$-8a
$4E^0$-8d	12	8	[3, 0, 0, 3], [1, 4, 0, 1], [5, 0, 0, 5], [1, 2, 2, 5], [1, 2, 0, 3], [1, 2, 0, 5]	$8(t^2 + 1)$	$2C^0$-2a
$4E^0$-8e	12	8	[3, 0, 0, 3], [1, 4, 0, 1], [5, 0, 0, 5], [1, 2, 2, 5], [1, 2, 0, 5], [0, 1, 1, 0]	$(t^2 + 1)/(t^2 - 2t - 1)$	$4C^0$-8b
$4E^0$-8f	12	8	[3, 0, 0, 3], [1, 4, 0, 1], [5, 0, 0, 5], [1, 2, 2, 5], [3, 0, 0, 5], [0, 1, 3, 2]	$(t^2 + 2t - 1)/(t^2 + 1)$	$2C^0$-8b
$4E^0$-8g	12	8	[3, 0, 0, 3], [1, 4, 0, 1], [5, 0, 0, 5], [1, 2, 2, 5], [3, 0, 0, 5], [2, 1, 1, 2]	$4t/(t^2 + 2)$	$2C^0$-8b
$4E^0$-8h	12	8	[3, 0, 0, 3], [1, 4, 0, 1], [5, 0, 0, 5], [1, 2, 2, 5], [3, 2, 0, 5], [0, 1, 3, 0]	$2(t^2 + 1)/(t^2 + 2t - 1)$	$2C^0$-8b
$4E^0$-8i	12	8	[3, 0, 0, 3], [1, 4, 0, 1], [5, 0, 0, 5], [1, 2, 2, 5], [3, 2, 0, 5], [2, 1, 1, 2]	$(t^2 + 2)/(2t)$	$2C^0$-8b

group	i	N	generators	map	super
4F^0-4a	12	4	[0, 1, 3, 0], [1, 0, 0, 3]	$8(t^2 - 1)$	4C^0-4a
4F^0-4b	12	4	[0, 1, 3, 0], [2, 1, 1, 2]	$8(t^2 + 1)$	4C^0-4a
4F^0-8a	12	8	[3, 0, 0, 3], [1, 4, 0, 1], [0, 3, 5, 0], [1, 0, 0, 3], [1, 2, 2, 1]	$4(t^2 + 2)$	4C^0-4a
4F^0-8b	12	8	[3, 0, 0, 3], [1, 4, 0, 1], [0, 3, 5, 0], [3, 0, 0, 5], [1, 2, 2, 1]	$4(t^2 - 2)$	4C^0-4a
4G^0-16a	24	16	[1, 4, 0, 1], [7, 0, 0, 7], [3, 0, 0, 11], [1, 0, 4, 1], [1, 1, 0, 5], [1, 5, 2, 5]	$(1 - t^2)/(2t)$	4E^0-8f
4G^0-4a	24	4	[3, 0, 0, 3], [1, 0, 0, 3]	$1/4t^2$	4E^0-4c
4G^0-4b	24	4	[3, 0, 0, 3], [1, 3, 0, 3]	$t^2/2$	4E^0-4a
4G^0-8a	24	8	[3, 0, 0, 3], [1, 4, 0, 1], [5, 0, 0, 5], [1, 0, 4, 1], [0, 1, 3, 0], [2, 1, 5, 2]	$4t/(t^2 - 2)$	4F^0-4b
4G^0-8b	24	8	[3, 0, 0, 3], [1, 4, 0, 1], [5, 0, 0, 5], [1, 0, 4, 1], [1, 0, 0, 3], [0, 1, 1, 0]	$2(t^2 + 2t + 2)/(t^2 - 2)$	4F^0-4a
4G^0-8c	24	8	[3, 0, 0, 3], [1, 4, 0, 1], [5, 0, 0, 5], [1, 0, 4, 1], [1, 0, 0, 3], [1, 2, 0, 5]	$t^2/2$	4E^0-4c
4G^0-8d	24	8	[3, 0, 0, 3], [1, 4, 0, 1], [5, 0, 0, 5], [1, 0, 4, 1], [1, 2, 0, 5], [3, 0, 2, 5]	$2(t^2 - 1)/(t^2 + 1)$	4F^0-8b
4G^0-8e	24	8	[3, 0, 0, 3], [1, 4, 0, 1], [5, 0, 0, 5], [1, 0, 4, 1], [1, 3, 0, 3], [1, 2, 0, 5]	t^2	4E^0-4a
4G^0-8f	24	8	[3, 0, 0, 3], [1, 4, 0, 1], [5, 0, 0, 5], [1, 0, 4, 1], [1, 3, 0, 3], [1, 3, 2, 3]	$4t/(t^2 + 2)$	4F^0-4a
8B^0-8a	12	8	[3, 0, 0, 3], [0, 3, 5, 0], [1, 2, 2, 5], [1, 0, 0, 3], [1, 0, 0, 5]	$16t^2$	4C^0-4a
8B^0-8b	12	8	[3, 0, 0, 3], [0, 3, 5, 0], [1, 2, 2, 5], [1, 0, 0, 3], [1, 4, 0, 5]	$32t^2$	4C^0-4a
8B^0-8c	12	8	[3, 0, 0, 3], [0, 3, 5, 0], [1, 2, 2, 5], [3, 2, 0, 1], [1, 0, 0, 5]	$32t^2$	4C^0-4b
8B^0-8d	12	8	[3, 0, 0, 3], [0, 3, 5, 0], [1, 2, 2, 5], [3, 2, 0, 1], [1, 4, 0, 5]	$16t^2$	4C^0-4b
8C^0-8a	12	8	[3, 0, 0, 3], [5, 0, 0, 5], [0, 3, 5, 2], [1, 2, 0, 3], [1, 0, 0, 5]	$-8t^2$	4B^0-4b
8C^0-8b	12	8	[3, 0, 0, 3], [5, 0, 0, 5], [0, 3, 5, 2], [1, 2, 0, 3], [1, 4, 0, 5]	$-4(t^2 + 4)$	4B^0-4b
8C^0-8c	12	8	[3, 0, 0, 3], [5, 0, 0, 5], [0, 3, 5, 2], [3, 2, 0, 1], [1, 0, 0, 5]	$-8(t^2 + 2)$	4B^0-4b
8C^0-8d	12	8	[3, 0, 0, 3], [5, 0, 0, 5], [0, 3, 5, 2], [3, 2, 0, 1], [1, 4, 0, 5]	$-t^2$	4B^0-4b
8D^0-8a	12	8	[2, 1, 3, 2], [0, 3, 5, 0], [1, 0, 0, 3], [1, 0, 0, 5]	$16/(t^2 - 2)$	4C^0-4a
8D^0-8b	12	8	[2, 1, 3, 2], [0, 3, 5, 0], [1, 0, 0, 3], [1, 4, 0, 5]	$32/(t^2 + 4)$	4C^0-4a
8D^0-8c	12	8	[2, 1, 3, 2], [0, 3, 5, 0], [1, 4, 0, 3], [1, 0, 0, 5]	$16/(t^2 + 2)$	4C^0-4a
8D^0-8d	12	8	[2, 1, 3, 2], [0, 3, 5, 0], [1, 4, 0, 3], [1, 4, 0, 5]	$32/(t^2 - 4)$	4C^0-4a
8E^0-16a	16	16	[3, 4, 0, 11], [2, 3, 3, 5], [3, 3, 0, 5], [0, 1, 3, 0]	$-4t/(t^2 + 2)$	4D^0-8a
8E^0-16b	16	16	[3, 4, 0, 11], [2, 3, 3, 5], [3, 3, 0, 5], [0, 3, 1, 0]	$-2(t^2 - 2t + 2)/(t^2 - 4t + 2)$	4D^0-8a
8F^0-8a	16	8	[1, 1, 1, 2], [0, 3, 5, 0], [3, 3, 0, 5], [2, 1, 1, 3]	$8(t^4 - 4t^2 - 8t - 4)/(t^2 - 2)^2$	4A^0-4a
8G^0-16a	24	16	[1, 2, 0, 1], [7, 0, 0, 7], [3, 0, 0, 11], [1, 0, 8, 1], [3, 1, 0, 5], [1, 2, 2, 1]	$(1 - t^2)/(2t)$	4E^0-8e

group	i	N	generators	map	supergroup
$8G^0$-8a	24	8	[1, 2, 0, 1], [3, 0, 0, 3], [5, 0, 0, 5], [1, 0, 0, 3], [1, 0, 0, 5]	$(t^2-1)/t$	$8C^0$-8b
$8G^0$-8b	24	8	[1, 2, 0, 1], [3, 0, 0, 3], [5, 0, 0, 5], [1, 0, 0, 3], [1, 0, 4, 5]	$(t^2+2)/(2t)$	$8C^0$-8a
$8G^0$-8c	24	8	[1, 2, 0, 1], [3, 0, 0, 3], [5, 0, 0, 5], [1, 0, 0, 3], [1, 1, 0, 5]	$(t^2-2)/(2t)$	$4E^0$-8c
$8G^0$-8d	24	8	[1, 2, 0, 1], [3, 0, 0, 3], [5, 0, 0, 5], [1, 0, 0, 5], [3, 0, 2, 1]	$t^2/2$	$4E^0$-4b
$8G^0$-8e	24	8	[1, 2, 0, 1], [3, 0, 0, 3], [5, 0, 0, 5], [1, 0, 2, 3], [3, 2, 2, 1]	t^2	$4E^0$-4b
$8G^0$-8f	24	8	[1, 2, 0, 1], [3, 0, 0, 3], [5, 0, 0, 5], [1, 1, 0, 3], [1, 0, 0, 5]	$(t^2-1)/(2t)$	$4E^0$-4a
$8G^0$-8g	24	8	[1, 2, 0, 1], [3, 0, 0, 3], [5, 0, 0, 5], [1, 1, 0, 3], [1, 1, 4, 1]	$(t^2+2)/(2t)$	$4E^0$-8i
$8G^0$-8h	24	8	[1, 2, 0, 1], [3, 0, 0, 3], [5, 0, 0, 5], [1, 1, 0, 3], [3, 2, 2, 1]	$2(t^2+1)/(t^2+2t-1)$	$4E^0$-8g
$8G^0$-8i	24	8	[1, 2, 0, 1], [3, 0, 0, 3], [5, 0, 0, 5], [1, 3, 2, 3], [3, 2, 2, 1]	$(t^2+2)/(2t)$	$4E^0$-8g
$8G^0$-8j	24	8	[1, 2, 0, 1], [3, 0, 0, 3], [5, 0, 0, 5], [3, 0, 0, 5], [1, 1, 4, 1]	$2(t^2+1)/(t^2-2t-1)$	$4E^0$-8i
$8G^0$-8k	24	8	[1, 2, 0, 1], [3, 0, 0, 3], [5, 0, 0, 5], [3, 1, 0, 5], [1, 0, 4, 5]	$2(t^2-1)/(t^2+1)$	$8D^0$-8a
$8G^0$-8l	24	8	[1, 2, 0, 1], [3, 0, 0, 3], [5, 0, 0, 5], [3, 1, 0, 5], [3, 0, 2, 1]	$(t^2-2)/(2t)$	$4E^0$-8a
$8H^0$-8a	24	8	[3, 0, 0, 3], [1, 4, 4, 1], [0, 3, 5, 0], [1, 0, 0, 3], [1, 0, 0, 5]	$2t/(t^2-2)$	$8B^0$-8a
$8H^0$-8b	24	8	[3, 0, 0, 3], [1, 4, 4, 1], [0, 3, 5, 0], [1, 0, 0, 3], [1, 2, 2, 3]	$4t/(t^2-2)$	$4F^0$-8a
$8H^0$-8c	24	8	[3, 0, 0, 3], [1, 4, 4, 1], [0, 3, 5, 0], [1, 0, 0, 3], [1, 4, 0, 5]	$t/(t^2+1)$	$8B^0$-8b
$8H^0$-8d	24	8	[3, 0, 0, 3], [1, 4, 4, 1], [0, 3, 5, 0], [1, 0, 0, 5], [1, 2, 2, 3]	$(t^2+1)/(t^2+2t-1)$	$8B^0$-8a
$8H^0$-8e	24	8	[3, 0, 0, 3], [1, 4, 4, 1], [0, 3, 5, 0], [1, 4, 0, 3], [1, 0, 0, 5]	$2t/(t^2+2)$	$8B^0$-8a
$8H^0$-8f	24	8	[3, 0, 0, 3], [1, 4, 4, 1], [0, 3, 5, 0], [1, 4, 0, 3], [1, 2, 2, 1]	$(t^2-2)/(2t)$	$4F^0$-8a
$8H^0$-8g	24	8	[3, 0, 0, 3], [1, 4, 4, 1], [0, 3, 5, 0], [1, 4, 0, 3], [1, 4, 0, 5]	$t/(t^2-1)$	$8B^0$-8b
$8H^0$-8h	24	8	[3, 0, 0, 3], [1, 4, 4, 1], [0, 3, 5, 0], [1, 4, 0, 5], [2, 1, 1, 2]	$(t^2-1)/(2t)$	$4F^0$-4b
$8H^0$-8i	24	8	[3, 0, 0, 3], [1, 4, 4, 1], [0, 3, 5, 0], [3, 0, 0, 5], [1, 2, 2, 1]	$(t^2+2)/(2t)$	$4F^0$-8b
$8H^0$-8j	24	8	[3, 0, 0, 3], [1, 4, 4, 1], [0, 3, 5, 0], [3, 0, 0, 5], [2, 3, 5, 2]	$2(t^2+1)/(t^2+2t-1)$	$4F^0$-8b
$8H^0$-8k	24	8	[3, 0, 0, 3], [1, 4, 4, 1], [0, 3, 5, 0], [3, 4, 0, 5], [1, 2, 2, 1]	$(t^2+2t-1)/(t^2+1)$	$4F^0$-8b
$8H^0$-8l	24	8	[3, 0, 0, 3], [1, 4, 4, 1], [0, 3, 5, 0], [3, 4, 0, 5], [2, 1, 1, 2]	$4t/(t^2+2)$	$4F^0$-8b
$8I^0$-8a	24	8	[7, 0, 0, 7], [0, 3, 5, 2], [1, 4, 0, 5], [1, 6, 0, 3]	$4t^2/(t^2-2)$	$8C^0$-8d
$8I^0$-8b	24	8	[7, 0, 0, 7], [0, 3, 5, 2], [3, 2, 0, 1], [1, 4, 0, 5]	$4t^2/(t^2+2)$	$8C^0$-8d
$8I^0$-8c	24	8	[7, 0, 0, 7], [0, 3, 5, 2], [3, 2, 0, 1], [5, 4, 0, 1]	$4/(t^2-1)$	$8C^0$-8d
$8I^0$-8d	24	8	[7, 0, 0, 7], [0, 3, 5, 2], [5, 2, 0, 3], [5, 4, 0, 1]	$4/(t^2+1)$	$8C^0$-8d
$8J^0$-8a	24	8	[3, 2, 0, 3], [5, 2, 0, 5], [1, 2, 4, 1], [1, 0, 0, 3], [1, 0, 0, 5]	$(t^2+2)/t^2$	$4E^0$-4c
$8J^0$-8b	24	8	[3, 2, 0, 3], [5, 2, 0, 5], [1, 2, 4, 1], [1, 0, 0, 3], [1, 2, 0, 5]	t^2-1	$4E^0$-4c
$8J^0$-8c	24	8	[3, 2, 0, 3], [5, 2, 0, 5], [1, 2, 4, 1], [1, 2, 0, 3], [1, 0, 0, 5]	$t^2/(t^2-2)$	$4E^0$-4c
$8J^0$-8d	24	8	[3, 2, 0, 3], [5, 2, 0, 5], [1, 2, 4, 1], [1, 2, 0, 3], [1, 2, 0, 5]	t^2+1	$4E^0$-4c

group	i	N	generators	map	super
8K^0-16a	24	16	[1, 4, 0, 1], [7, 0, 0, 7], [0, 3, 5, 0], [3, 0, 0, 5], [1, 2, 2, 1]	$(t^2+4)/2$	4F^0-8b
8K^0-16b	24	16	[1, 4, 0, 1], [7, 0, 0, 7], [0, 3, 5, 0], [7, 0, 0, 9], [1, 2, 2, 1]	$(t^2-4)/2$	4F^0-8b
8K^0-16c	24	16	[1, 4, 0, 1], [7, 0, 0, 7], [2, 3, 1, 2], [3, 0, 0, 5], [1, 2, 2, 1]	t^2+2	4F^0-8b
8K^0-16d	24	16	[1, 4, 0, 1], [7, 0, 0, 7], [2, 3, 1, 2], [7, 0, 0, 9], [1, 2, 2, 1]	t^2-2	4F^0-8b
8L^0-8a	24	8	[0, 3, 5, 0], [5, 2, 2, 1], [1, 2, 0, 3], [1, 4, 0, 5]	$(t^2-2t-1)/(4t)$	8B^0-8d
8L^0-8b	24	8	[0, 3, 5, 0], [5, 2, 2, 1], [5, 4, 0, 1], [3, 6, 0, 1]	$2t/(t^2-2t-1)$	8B^0-8d
8N^0-16a	48	16	[1, 4, 0, 1], [7, 0, 0, 7], [9, 0, 0, 9], [1, 0, 4, 1], [0, 1, 1, 0], [2, 3, 5, 6]	$(t^2+2)/(2t)$	4G^0-8a
8N^0-16b	48	16	[1, 4, 0, 1], [7, 0, 0, 7], [9, 0, 0, 9], [1, 0, 4, 1], [0, 1, 5, 0], [2, 5, 1, 2]	$t^2/2$	4G^0-8e
8N^0-16c	48	16	[1, 4, 0, 1], [7, 0, 0, 7], [9, 0, 0, 9], [1, 0, 4, 1], [1, 2, 2, 1], [0, 5, 1, 0]	$4t/(t^2-1)$	8K^0-16c
8N^0-16d	48	16	[1, 4, 0, 1], [7, 0, 0, 7], [9, 0, 0, 9], [1, 0, 4, 1], [3, 2, 2, 5], [2, 3, 5, 6]	$2(t^2+1)/(t^2-2t-1)$	4G^0-8a
8N^0-16e	48	16	[1, 4, 0, 1], [7, 0, 0, 7], [9, 0, 0, 9], [1, 0, 4, 1], [7, 0, 0, 9], [1, 2, 2, 1]	$(t^2-1)/(2t)$	4G^0-8d
8N^0-16f	48	16	[1, 4, 0, 1], [7, 0, 0, 7], [9, 0, 0, 9], [1, 0, 4, 1], [7, 0, 0, 9], [2, 1, 1, 6]	$(t^2-2)/(2t)$	4G^0-8f
8N^0-32a	48	32	[1, 4, 0, 1], [15, 0, 0, 15], [7, 0, 0, 23], [1, 0, 4, 1], [3, 0, 0, 5], [2, 3, 1, 0]	$(1-2t-t^2)/(t^2-2t-1)$	4G^0-16a
8N^0-8a	48	8	[1, 4, 0, 1], [7, 0, 0, 7], [1, 0, 4, 1], [1, 2, 0, 3], [1, 0, 0, 5]	t^2	4G^0-4a
8N^0-8b	48	8	[1, 4, 0, 1], [7, 0, 0, 7], [1, 0, 4, 1], [1, 2, 0, 3], [1, 2, 0, 5]	$(t^2-2)/(2t)$	4G^0-8c
8N^0-8c	48	8	[1, 4, 0, 1], [7, 0, 0, 7], [1, 0, 4, 1], [1, 2, 0, 3], [3, 2, 0, 5]	$(t^2+1)/t$	4G^0-4a
8N^0-8d	48	8	[1, 4, 0, 1], [7, 0, 0, 7], [1, 0, 4, 1], [3, 0, 0, 1], [1, 0, 2, 5]	$2(t^2+1)/(t^2+2t-1)$	4G^0-8c
8N^0-8e	48	8	[1, 4, 0, 1], [7, 0, 0, 7], [1, 0, 4, 1], [3, 0, 0, 1], [1, 2, 0, 5]	$(t^2+2)/(2t)$	4G^0-8c
8N^0-8f	48	8	[1, 4, 0, 1], [7, 0, 0, 7], [1, 0, 4, 1], [3, 0, 0, 1], [5, 0, 0, 1]	$(t^2-1)/t$	4G^0-4a
8O^0-16a	48	16	[1, 4, 0, 1], [7, 0, 0, 7], [3, 2, 0, 11], [1, 0, 8, 1], [1, 3, 0, 5], [5, 1, 4, 3]	$(1-t^2)/(2t)$	8G^0-8j
8O^0-8a	48	8	[3, 2, 0, 3], [5, 2, 0, 5], [1, 0, 0, 3], [1, 0, 4, 5]	$(t^2-2)/(2t)$	8G^0-8b
8O^0-8b	48	8	[3, 2, 0, 3], [5, 2, 0, 5], [1, 0, 0, 5], [1, 0, 4, 3]	$(t^2-1)/(2t)$	8G^0-8a
8O^0-8c	48	8	[3, 2, 0, 3], [5, 2, 0, 5], [1, 2, 0, 3], [1, 0, 0, 5]	$2t/(t^2-1)$	8J^0-8a
8O^0-8d	48	8	[3, 2, 0, 3], [5, 2, 0, 5], [1, 2, 0, 3], [1, 2, 0, 5]	$(t^2+2)/(2t)$	8J^0-8b
8O^0-8e	48	8	[3, 2, 0, 3], [5, 2, 0, 5], [1, 2, 0, 5], [1, 2, 4, 3]	$(t^2+1)/(2t)$	8I^0-8a
8O^0-8f	48	8	[3, 2, 0, 3], [5, 2, 0, 5], [1, 3, 0, 3], [1, 0, 0, 5]	$t^2/2$	8G^0-8f
8O^0-8g	48	8	[3, 2, 0, 3], [5, 2, 0, 5], [1, 3, 0, 3], [1, 2, 0, 5]	t^2	8G^0-8f
8O^0-8h	48	8	[3, 2, 0, 3], [5, 2, 0, 5], [1, 3, 0, 3], [1, 3, 4, 1]	$(t^2-2)/(2t)$	8G^0-8g
8O^0-8i	48	8	[3, 2, 0, 3], [5, 2, 0, 5], [3, 0, 0, 5], [1, 2, 4, 3]	$(t^2-2)/(t^2-4t+2)$	8J^0-8a
8O^0-8j	48	8	[3, 2, 0, 3], [5, 2, 0, 5], [3, 2, 0, 5], [1, 0, 4, 3]	$(t^2+2)/(2t)$	8G^0-8b
8O^0-8k	48	8	[3, 2, 0, 3], [5, 2, 0, 5], [3, 3, 0, 5], [1, 0, 4, 3]	$(t^2+4t+2)/(t^2-2)$	8I^0-8b
8O^0-8l	48	8	[3, 2, 0, 3], [5, 2, 0, 5], [3, 3, 0, 5], [1, 1, 4, 1]	$(t^2+2)/(2t)$	8G^0-8c
8P^0-8a	48	8	[3, 4, 4, 3], [0, 3, 5, 0], [3, 4, 0, 1], [1, 4, 0, 5]	$(t^2+2)/(t^2-4t+2)$	8H^0-8g
8P^0-8b	48	8	[3, 4, 4, 3], [0, 3, 5, 0], [3, 4, 0, 1], [5, 4, 0, 1]	$(t^2+1)/(2(t-1))$	8H^0-8g

group	i	N	generators	map	supergroup
$16B^0$-16a	24	16	[3, 0, 0, 11], [0, 3, 5, 0], [1, 2, 2, 5], [1, 0, 0, 5], [1, 8, 0, 3]	$t^2/2$	$8B^0$-8a
$16B^0$-16b	24	16	[3, 0, 0, 11], [0, 3, 5, 0], [1, 2, 2, 5], [1, 4, 0, 5], [5, 2, 0, 3]	$t^2/2$	$8B^0$-8d
$16B^0$-16c	24	16	[3, 0, 0, 11], [0, 3, 5, 0], [1, 2, 2, 5], [3, 0, 0, 5], [1, 8, 0, 3]	t^2	$8B^0$-8a
$16B^0$-16d	24	16	[3, 0, 0, 11], [0, 3, 5, 0], [1, 2, 2, 5], [3, 6, 0, 5], [3, 4, 0, 7]	t^2	$8B^0$-8d
$16C^0$-16a	24	16	[7, 0, 0, 7], [3, 8, 0, 11], [0, 3, 5, 2], [1, 2, 0, 3], [1, 4, 0, 5]	$t^2/2$	$8C^0$-8b
$16C^0$-16b	24	16	[7, 0, 0, 7], [3, 8, 0, 11], [0, 3, 5, 2], [1, 2, 0, 3], [5, 4, 0, 1]	t^2	$8C^0$-8b
$16C^0$-16c	24	16	[7, 0, 0, 7], [3, 8, 0, 11], [0, 3, 5, 2], [1, 4, 0, 5], [5, 2, 0, 3]	$2t^2$	$8C^0$-8d
$16C^0$-16d	24	16	[7, 0, 0, 7], [3, 8, 0, 11], [0, 3, 5, 2], [5, 4, 0, 1], [1, 6, 0, 3]	t^2	$8C^0$-8d
$16D^0$-16a	24	16	[7, 0, 0, 7], [3, 0, 0, 11], [0, 3, 5, 2], [1, 4, 0, 5], [5, 2, 0, 3]	$t^2 - 4$	$8C^0$-8d
$16D^0$-16b	24	16	[7, 0, 0, 7], [3, 0, 0, 11], [0, 3, 5, 2], [3, 2, 0, 1], [1, 4, 0, 5]	$t^2 + 4$	$8C^0$-8d
$16D^0$-16c	24	16	[7, 0, 0, 7], [3, 0, 0, 11], [0, 3, 5, 2], [3, 2, 0, 1], [5, 2, 0, 3]	$2(t^2 - 2)$	$8C^0$-8d
$16D^0$-16d	24	16	[7, 0, 0, 7], [3, 0, 0, 11], [0, 3, 5, 2], [3, 6, 0, 5], [3, 4, 0, 7]	$2(t^2 + 2)$	$8C^0$-8d
$16E^0$-16a	24	16	[0, 3, 5, 8], [2, 1, 3, 10], [1, 0, 0, 3], [1, 0, 0, 5]	$(t^2 + 4)/2$	$8D^0$-8a
$16E^0$-16b	24	16	[0, 3, 5, 8], [2, 1, 3, 10], [1, 0, 0, 3], [1, 8, 0, 5]	$t^2 - 2$	$8D^0$-8a
$16E^0$-16c	24	16	[0, 3, 5, 8], [2, 1, 3, 10], [1, 0, 0, 5], [1, 8, 0, 3]	$(t^2 - 4)/2$	$8D^0$-8a
$16E^0$-16d	24	16	[0, 3, 5, 8], [2, 1, 3, 10], [3, 0, 0, 5], [1, 8, 0, 3]	$t^2 + 2$	$8D^0$-8a
$16F^0$-32a	32	32	[3, 4, 0, 11], [6, 3, 7, 9], [3, 3, 0, 5], [0, 3, 1, 0]	$-(t^2 + 2)/(2t)$	$8E^0$-16b
$16F^0$-32b	32	32	[3, 4, 0, 11], [6, 3, 7, 9], [5, 5, 0, 3], [0, 3, 1, 0]	$(t^2 - 4t + 2)/(t^2 - 2t + 2)$	$8E^0$-16b
$16G^0$-16a	48	16	[1, 2, 0, 1], [7, 0, 0, 7], [3, 0, 8, 11], [1, 3, 2, 3], [3, 2, 2, 1]	$2(t^2 + 1)/(t^2 + 2t - 1)$	$8G^0$-8i
$16G^0$-16b	48	16	[1, 2, 0, 1], [7, 0, 0, 7], [3, 0, 8, 11], [3, 0, 0, 1], [1, 0, 0, 5]	$t^2/2$	$8G^0$-8a
$16G^0$-16c	48	16	[1, 2, 0, 1], [7, 0, 0, 7], [3, 0, 8, 11], [3, 0, 0, 1], [5, 0, 0, 1]	t^2	$8G^0$-8a
$16G^0$-16d	48	16	[1, 2, 0, 1], [7, 0, 0, 7], [3, 0, 8, 11], [3, 0, 0, 1], [5, 1, 0, 1]	$(t^2 - 2)/(2t)$	$8G^0$-8c
$16G^0$-16e	48	16	[1, 2, 0, 1], [7, 0, 0, 7], [3, 0, 8, 11], [3, 1, 0, 1], [1, 1, 0, 5]	$(t^2 + 2)/(2t)$	$8G^0$-8g
$16G^0$-16f	48	16	[1, 2, 0, 1], [7, 0, 0, 7], [3, 0, 8, 11], [3, 1, 0, 1], [1, 1, 4, 1]	$2(t^2 + 1)/(t^2 + 2t - 1)$	$8G^0$-8g
$16G^0$-16g	48	16	[1, 2, 0, 1], [7, 0, 0, 7], [3, 0, 8, 11], [3, 1, 0, 1], [5, 0, 0, 1]	$(t^2 - 1)/(2t)$	$8G^0$-8f
$16G^0$-16h	48	16	[1, 2, 0, 1], [7, 0, 0, 7], [3, 0, 8, 11], [3, 2, 2, 1], [1, 4, 2, 3]	t^2	$8G^0$-8e
$16G^0$-16i	48	16	[1, 2, 0, 1], [7, 0, 0, 7], [3, 0, 8, 11], [3, 2, 2, 1], [3, 3, 4, 5]	$(t^2 + 2)/(2t)$	$8G^0$-8i
$16G^0$-16j	48	16	[1, 2, 0, 1], [7, 0, 0, 7], [3, 0, 8, 11], [3, 4, 2, 1], [1, 2, 4, 5]	$t^2/2$	$8G^0$-8e
$16G^0$-16k	48	16	[1, 2, 0, 1], [7, 0, 0, 7], [3, 0, 8, 11], [7, 1, 0, 9], [1, 2, 4, 5]	$(t^2 - 1)/(2t)$	$8G^0$-8k
$16G^0$-16l	48	16	[1, 2, 0, 1], [7, 0, 0, 7], [3, 0, 8, 11], [7, 1, 0, 9], [3, 0, 2, 1]	$(t^2 - 2)/(2t)$	$8G^0$-8l
$16G^0$-32a	48	32	[1, 2, 0, 1], [15, 0, 0, 15], [7, 0, 0, 23], [3, 0, 8, 11], [3, 1, 0, 5], [5, 2, 2, 5]	$(1 - 2t - t^2)/(t^2 - 2t - 1)$	$8G^0$-16a

group	i	N	generators	map	supergroup
$16H^0$-16a	48	16	[7, 0, 0, 7], [9, 0, 0, 9], [0, 3, 5, 2], [1, 4, 0, 5], [1, 2, 0, 7]	$4t/(t^2+2)$	$8I^0$-8a
$16H^0$-16b	48	16	[7, 0, 0, 7], [9, 0, 0, 9], [0, 3, 5, 2], [1, 4, 0, 5], [1, 6, 0, 3]	$2(t^2+1)/(t^2-2t-1)$	$8I^0$-8a
$16H^0$-16c	48	16	[7, 0, 0, 7], [9, 0, 0, 9], [0, 3, 5, 2], [1, 4, 0, 5], [5, 2, 0, 3]	$4t/(t^2-2)$	$8I^0$-8b
$16H^0$-16d	48	16	[7, 0, 0, 7], [9, 0, 0, 9], [0, 3, 5, 2], [1, 6, 0, 3], [1, 2, 0, 7]	$(t^2+2t-1)/(t^2+1)$	$8I^0$-8a
$16H^0$-16e	48	16	[7, 0, 0, 7], [9, 0, 0, 9], [0, 3, 5, 2], [3, 2, 0, 1], [1, 2, 0, 7]	$(t^2-2)/(2t)$	$16C^0$-16c
$16H^0$-16f	48	16	[7, 0, 0, 7], [9, 0, 0, 9], [0, 3, 5, 2], [3, 2, 0, 1], [5, 2, 0, 3]	$(t^2-2)/(2t)$	$8I^0$-8b
$16H^0$-16g	48	16	[7, 0, 0, 7], [9, 0, 0, 9], [0, 3, 5, 2], [3, 2, 0, 1], [5, 4, 0, 1]	$(t^2+1)/t$	$16C^0$-16d
$16H^0$-16h	48	16	[7, 0, 0, 7], [9, 0, 0, 9], [0, 3, 5, 2], [3, 4, 0, 7], [5, 6, 0, 7]	$(t^2+2)/(2t)$	$16C^0$-16c
$16H^0$-16i	48	16	[7, 0, 0, 7], [9, 0, 0, 9], [0, 3, 5, 2], [5, 2, 0, 3], [1, 6, 0, 3]	$(t^2+2t-1)/(t^2+1)$	$16C^0$-16c
$16H^0$-16j	48	16	[7, 0, 0, 7], [9, 0, 0, 9], [0, 3, 5, 2], [5, 2, 0, 3], [5, 4, 0, 1]	$2t/(t^2-1)$	$8I^0$-8d
$16H^0$-16k	48	16	[7, 0, 0, 7], [9, 0, 0, 9], [0, 3, 5, 2], [5, 4, 0, 1], [1, 2, 0, 7]	$(t^2-1)/t$	$16C^0$-16d
$16H^0$-16l	48	16	[7, 0, 0, 7], [9, 0, 0, 9], [0, 3, 5, 2], [7, 2, 0, 5], [7, 4, 0, 3]	$(t^2+2)/(2t)$	$8I^0$-8a
$32A^0$-32a	48	32	[3, 8, 0, 11], [5, 8, 0, 13], [0, 7, 9, 2], [1, 4, 0, 5], [5, 6, 0, 3]	$2/t^2$	$16C^0$-16a
$32A^0$-32b	48	32	[3, 8, 0, 11], [5, 8, 0, 13], [0, 7, 9, 2], [5, 2, 0, 3], [5, 4, 0, 1]	t^2	$16C^0$-16d
$32A^0$-32c	48	32	[3, 8, 0, 11], [5, 8, 0, 13], [0, 7, 9, 2], [7, 2, 0, 1], [9, 2, 0, 3]	t^2	$16C^0$-16a
$32A^0$-32d	48	32	[3, 8, 0, 11], [5, 8, 0, 13], [0, 7, 9, 2], [7, 2, 0, 5], [7, 4, 0, 3]	$2/t^2$	$16C^0$-16d

group	i	N	generators	curve	map	supergroup
$16C^1$-16c	24	16	[2, 1, 3, 2], [0, 3, 5, 8], [1, 0, 0, 5], [1, 8, 0, 3]	256a2	$(x-1)/2$	$8D^0$-8a
$16C^1$-16d	24	16	[2, 1, 3, 2], [0, 3, 5, 8], [3, 0, 0, 5], [1, 8, 0, 3]	256a1	$x+1$	$8D^0$-8a
$16B^1$-16a	24	16	[3, 0, 0, 11], [0, 3, 5, 0], [2, 3, 9, 6], [1, 0, 0, 3], [1, 0, 0, 5]	256b2	$x/4$	$8B^0$-8a
$16B^1$-16c	24	16	[3, 0, 0, 11], [0, 3, 5, 0], [2, 3, 9, 6], [1, 4, 0, 3], [1, 0, 0, 5]	256b1	$x/2$	$8B^0$-8a
$16I^1$-16d	48	16	[3, 0, 0, 11], [0, 3, 5, 8], [1, 4, 12, 1], [1, 0, 0, 5], [1, 8, 0, 3]	256a1	$x-1$	$8H^0$-8a
$16I^1$-16f	48	16	[3, 0, 0, 11], [0, 3, 5, 8], [1, 4, 12, 1], [1, 8, 0, 3], [2, 3, 5, 2]	256a2	$\frac{4x-4y+12}{x^2-2x-15}$	$8H^0$-8b
$16I^1$-16g	48	16	[3, 0, 0, 11], [0, 3, 5, 8], [1, 4, 12, 1], [1, 8, 0, 5], [1, 2, 10, 3]	256a2	$\frac{2(y-x-1)}{x^2-2x-11}$	$8H^0$-8d
$16I^1$-16h	48	16	[3, 0, 0, 11], [0, 3, 5, 8], [1, 4, 12, 1], [1, 8, 0, 7], [1, 2, 10, 7]	256a1	$1/x$	$8H^0$-8j
$16I^1$-16j	48	16	[3, 0, 0, 11], [0, 3, 5, 8], [1, 4, 12, 1], [3, 0, 0, 5], [1, 8, 0, 3]	256a2	$(x+3)/2$	$8H^0$-8a
$16I^1$-16k	48	16	[3, 0, 0, 11], [0, 3, 5, 8], [1, 4, 12, 1], [3, 0, 0, 5], [2, 3, 5, 2]	256a2	$\frac{-x+1}{x+3}$	$8H^0$-8j
$8H^1$-16b	48	16	[7, 0, 0, 7], [1, 8, 0, 1], [1, 4, 4, 1], [0, 3, 5, 0], [3, 0, 0, 5], [1, 2, 2, 9]	256a2	$(x+3)/2$	$8H^0$-8i
$8H^1$-16c	48	16	[7, 0, 0, 7], [1, 8, 0, 1], [1, 4, 4, 1], [0, 3, 5, 0], [3, 0, 0, 5], [1, 2, 6, 9]	256a2	$\frac{x-1}{x+3}$	$8H^0$-8j
$8H^1$-16e	48	16	[7, 0, 0, 7], [1, 8, 0, 1], [1, 4, 4, 1], [0, 3, 5, 0], [7, 0, 0, 9], [1, 2, 2, 1]	256a1	$x-1$	$8H^0$-8i
$8H^1$-16g	48	16	[7, 0, 0, 7], [1, 8, 0, 1], [1, 4, 4, 1], [0, 3, 5, 0], [7, 0, 0, 9], [2, 3, 5, 2]	256a1	$-1/x$	$8H^0$-8j
$16D^1$-16d	24	16	[3, 8, 0, 11], [0, 3, 5, 0], [5, 2, 2, 1], [3, 2, 0, 5], [5, 4, 0, 1]	128a1	$(x+1)/2$	$8B^0$-8d
$8H^1$-16j	48	16	[7, 0, 0, 7], [1, 8, 0, 1], [1, 4, 4, 1], [0, 3, 5, 0], [7, 4, 0, 9], [1, 2, 2, 9]	256a2	$\frac{2(x+y+1)}{x^2-2x-11}$	$8H^0$-8k
$8D^1$-16b	24	16	[7, 0, 0, 7], [3, 4, 0, 11], [0, 3, 5, 0], [3, 0, 0, 5], [1, 2, 2, 9]	256a1	$x+1$	$4F^0$-8b
$8H^1$-16k	48	16	[7, 0, 0, 7], [1, 8, 0, 1], [1, 4, 4, 1], [0, 3, 5, 0], [7, 4, 0, 9], [1, 2, 6, 9]	256a2	$\frac{4(x-y+3)}{x^2-2x-15}$	$8H^0$-8l
$8D^1$-16c	24	16	[7, 0, 0, 7], [3, 4, 0, 11], [0, 3, 5, 0], [3, 4, 0, 5], [1, 2, 2, 1]	256a2	$(x-1)/2$	$4F^0$-8b
$16J^1$-16e	48	16	[7, 0, 0, 7], [0, 3, 5, 0], [5, 2, 2, 1], [1, 6, 0, 7], [7, 4, 0, 3]	128a2	$\frac{x^2+2x-7}{8x-8}$	$8B^0$-8d
$16J^1$-16g	48	16	[7, 0, 0, 7], [0, 3, 5, 0], [5, 2, 2, 1], [5, 4, 0, 1], [3, 6, 0, 1]	128a2	$(1-x)/2$	$8L^0$-8b
$16F^1$-16a	48	16	[3, 0, 0, 11], [1, 4, 4, 1], [0, 3, 5, 0], [1, 0, 0, 3], [1, 0, 0, 5]	256b1	$-2/x$	$8H^0$-8a
$16F^1$-16c	48	16	[3, 0, 0, 11], [1, 4, 4, 1], [0, 3, 5, 0], [1, 0, 0, 5], [1, 2, 2, 3]	256b2	$\frac{x^2+2y-8}{x^2-8x+8}$	$16B^0$-16c
$16F^1$-16d	48	16	[3, 0, 0, 11], [1, 4, 4, 1], [0, 3, 5, 0], [1, 4, 0, 3], [1, 0, 0, 5]	256b2	$4/x$	$8H^0$-8e
$16F^1$-16h	48	16	[3, 0, 0, 11], [1, 4, 4, 1], [0, 3, 5, 0], [3, 0, 0, 5], [1, 2, 2, 1]	256b1	x	$8H^0$-8i
$16F^1$-16j	48	16	[3, 0, 0, 11], [1, 4, 4, 1], [0, 3, 5, 0], [3, 4, 0, 1], [1, 2, 2, 1]	256b2	$x/2$	$8H^0$-8f
$16F^1$-16k	48	16	[3, 0, 0, 11], [1, 4, 4, 1], [0, 3, 5, 0], [3, 4, 0, 5], [1, 2, 2, 1]	256b2	$\frac{x^2+2y-8}{x^2+4x+8}$	$16B^0$-16a
$11C^1$-11a	55	11	[3, 4, 4, 2], [3, 1, 1, 8], [6, 4, 0, 5]	121b1	$J_{11}(x, y)$	$1A^0$-1a

The map $J_{11}(x,y)$ is due to Halberstadt [Hal98], and is defined by

$$J_{11}(x,y) := \frac{(f_1 f_2 f_3 f_4)^3}{f_5^2 f_6^{11}},$$

where

$$\begin{aligned}
f_1 &:= x^2 + 3x - 6,\\
f_2 &:= 11(x^2 - 5)y + (2x^4 + 23x^3 - 72x^2 - 28x + 127),\\
f_3 &:= 6y + 11x - 19,\\
f_4 &:= 22(x-2)y + (5x^3 + 17x^2 - 112x + 120),\\
f_5 &:= 11y + (2x^2 + 17x - 34),\\
f_6 &:= (x-4)y - (5x - 9).
\end{aligned}$$

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