Combinatorial Reconfiguration, BIRS, 25 January 2017

Kempe equivalence in regular graphs

Matthew Johnson
Durham University

Kempe Chains

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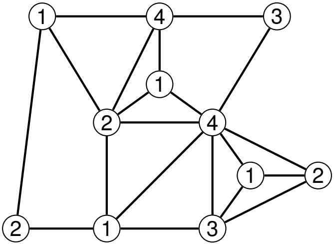

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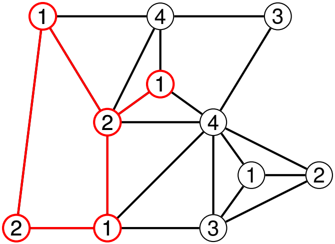

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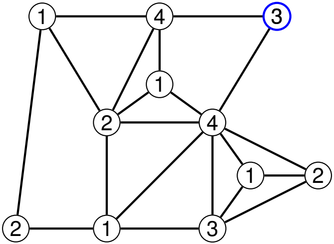

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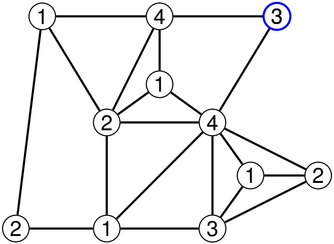

A (1,3)-component

These components are called Kempe chains.

Kempe Changes

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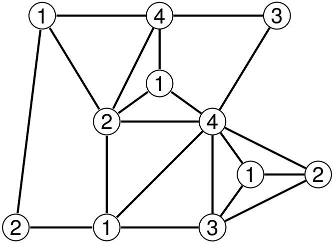

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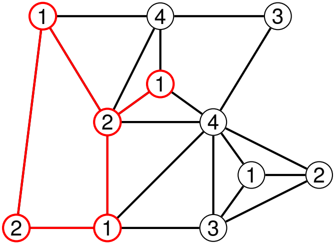

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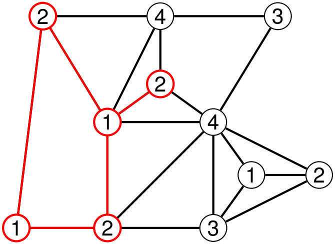

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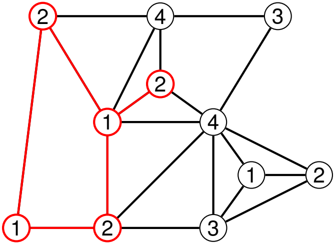

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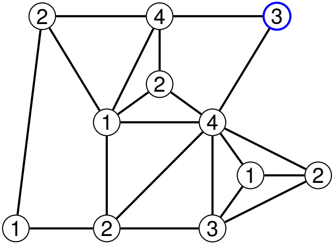

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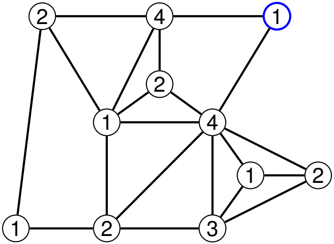

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A graph is d-degenerate if every induced subgraph has a vertex of degree at most d.

Theorem (Las Vergnas, Meyniel 1981)

Let k be greater than d. Then the set of k-colourings of a d-degenerate graph form a Kempe class.

Proof

Suppose instead that $G + v$ is the smallest d-degenerate graph with a pair of non-Kempe-equivalent k-colourings α and β, where v is a vertex of degree at most d.

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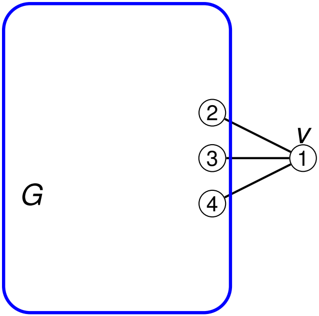

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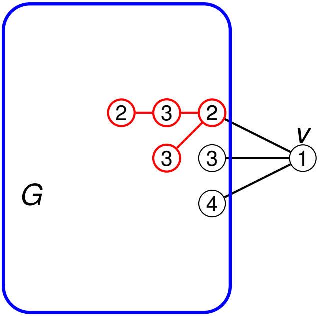

Kempe chain might not use the colour of v.

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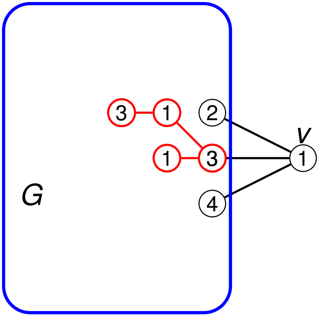

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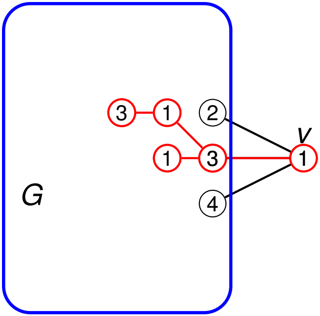

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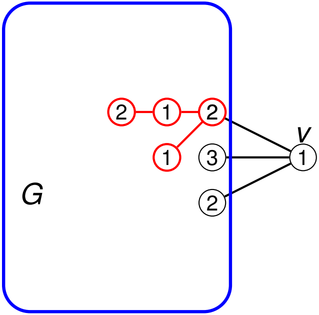

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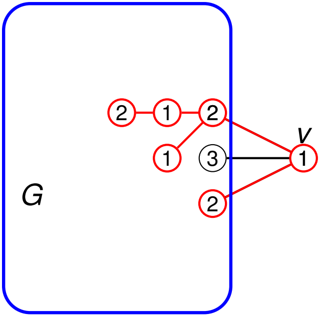

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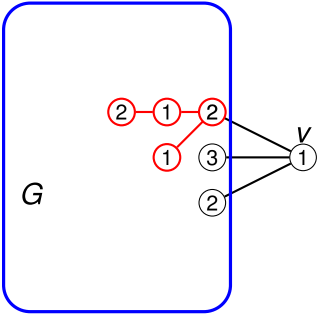

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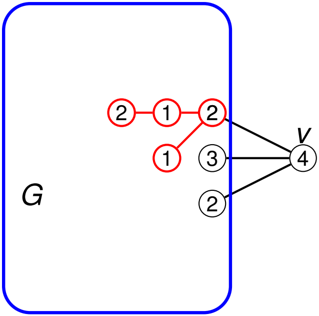

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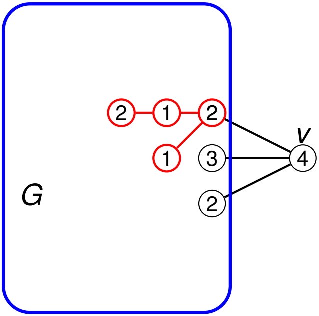

Kempe chain might use the colour of v and a colour that appears on more than one neighbour.

Then first change the colour of v.

If needed, make a final trivial change to v.

Regular Graphs

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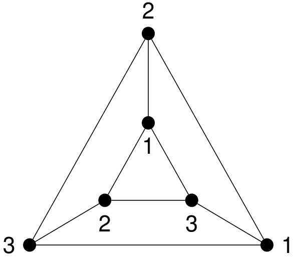

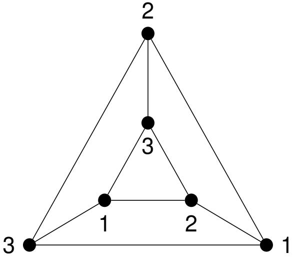

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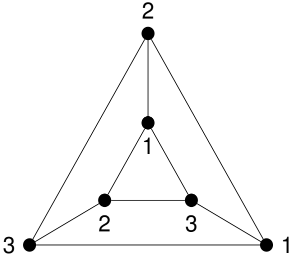

Observe that no Kempe change alters the colour partition, but that these differ.

Regular Graphs

Theorem (Bonamy, Bousquet, Feghali, J, Paulusma 2017)

Let $k \geq 3$. If G is a connected k-regular graph that is neither complete nor the triangular prism, then the k-colourings of G form a Kempe class.

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A useful result: the clique cutset lemma.

Lemma (Las Vergnas, Meyniel 1981)

Let k be a positive integer. Let G_1 and G_2 be two graphs such that $G_1 \cap G_2$ is complete. If the k-colourings of each of G_1 and G_2 form a Kempe class, then the k-colourings of $G_1 \cup G_2$ form a Kempe class.

k-Regular Graphs that are not 3-connected

If G is not 3-connected it has a cutset C of size 1 or 2. If this is a clique, then apply the clique cutset lemma (and then notice that the union of C and each connected component of $G - C$ is $(k-1)$-degenerate).

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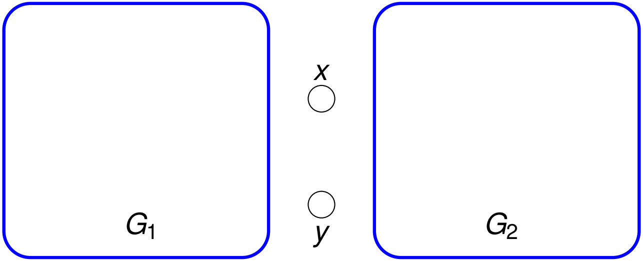
Otherwise the cutset is a pair of non-adjacent vertices.



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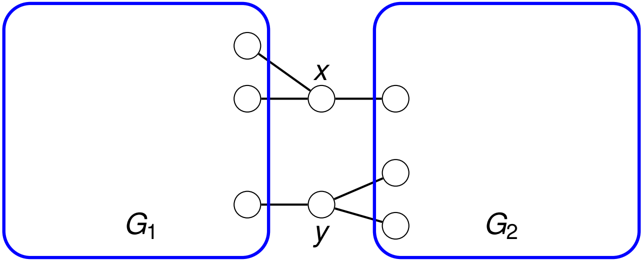

We can assume that x has more than one neighbour in G_1 and y has more than one neighbour in G_2.

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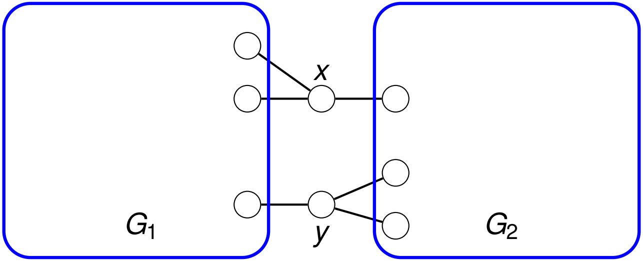

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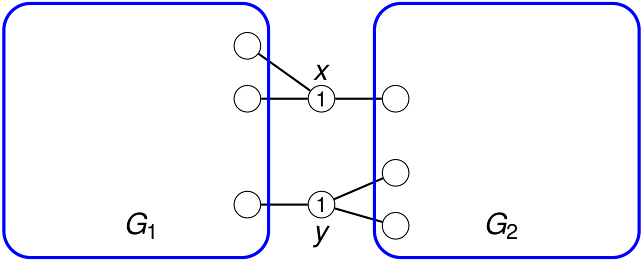

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Just to need that when x and y are coloured alike we can apply Kempe changes until they differ.

k-Regular Graphs that are 3-connected

Lemma

Let $k \geq 4$ be a positive integer.
Let G be a 3-connected non-complete k-regular graph.
Let u and v be two vertices of G that are not adjacent.
If there is a pair w_1 and w_2 of non-adjacent neighbours of v neither of which is adjacent to u, then the k-colourings of G are a Kempe class.

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The k-colourings of 3-connected non-complete k-regular graphs of diameter at least 3 form a Kempe class as a good pair can always be found.

Matching Lemma

Lemma

Let $k \geq 3$ be a positive integer.
Let G be a 3-connected non-complete k-regular graph.
Let u and v be two vertices with a common neighbour of G that are not adjacent.
If a pair of k-colourings of G can each be changed by a sequence of Kempe changes into a k-colouring where u and v are coloured alike, then the two k-colourings are Kempe equivalent.

k-Regular 3-connected Graphs of diameter 2

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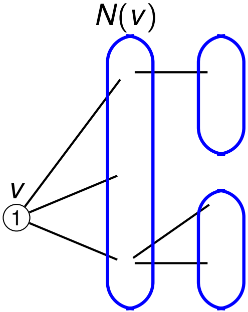

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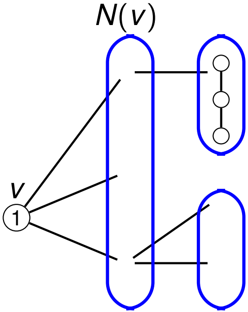

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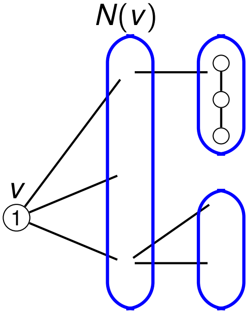

If the second neighbourhood contains an induced path, we can find a good pair.
So the second neighbourhood contains disjoint cliques.

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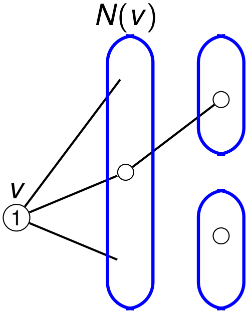

If a vertex in $N(v)$ neighbours one clique but not another, then we can find a good pair.

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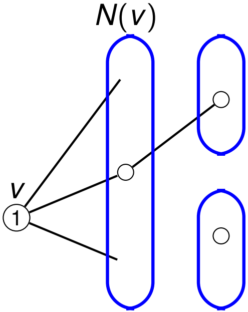

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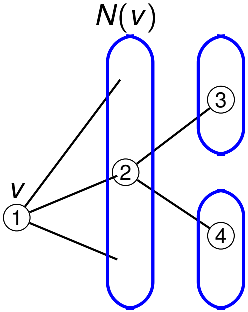

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The Matching Lemma is used if there is more than one clique.

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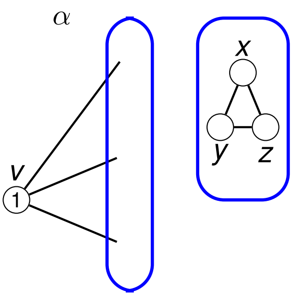

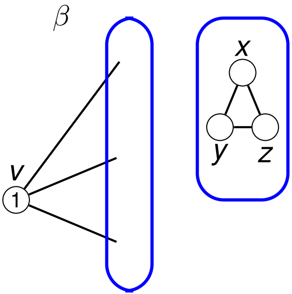

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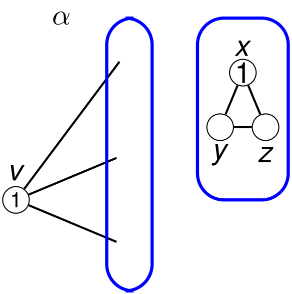

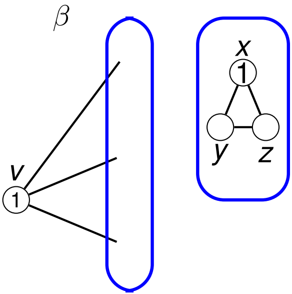

If the colour 1 appears on the same vertex on the clique (or not at all), use the Matching Lemma

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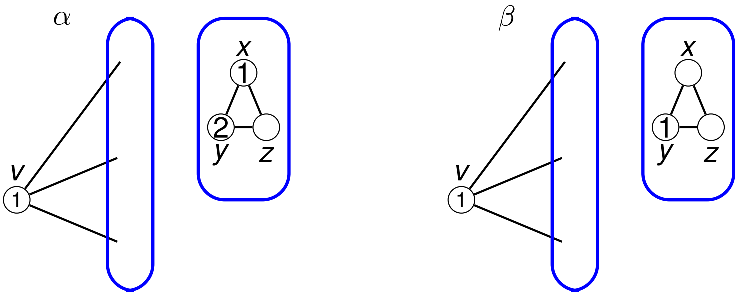

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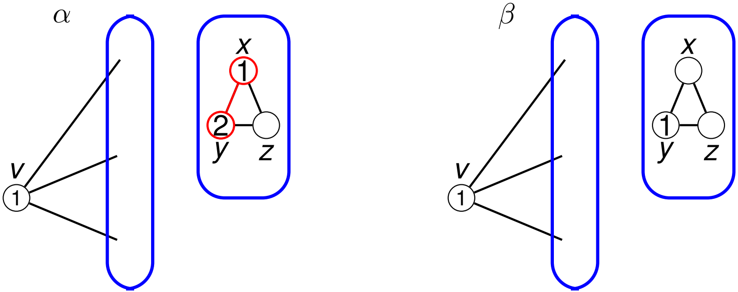

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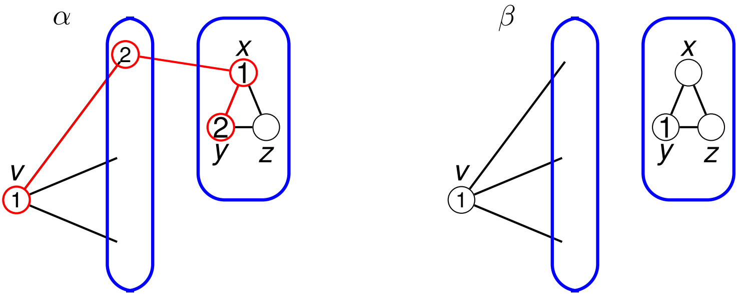

So the colour 1 appears on distinct vertices.
Match the colours of y with a single Kempe change.
Unless the Kempe chain includes v.

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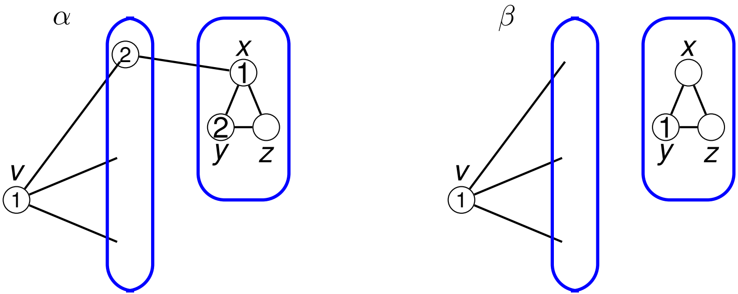

But then x has exactly one neighbour with each other colour.
And so x and z form a Kempe chain.

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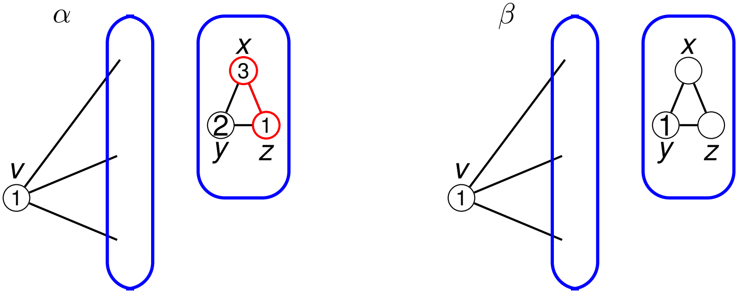

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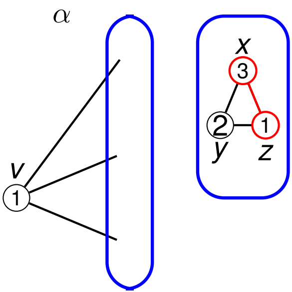

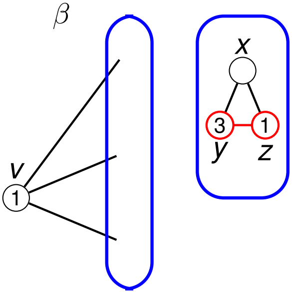

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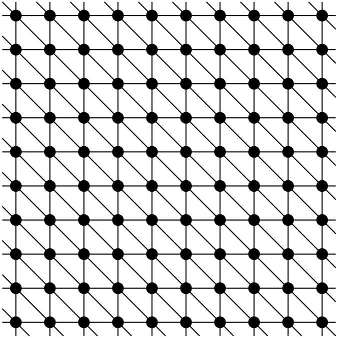
So we can apply the Matching Lemma using v and z.

Open Problems



Do the 5-colourings of a toroidal triangular lattice form a Kempe class? (Would prove the validity of WSK algorithm for simulating the antiferromagnetic Potts model.)

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What is the "distance" between k-colourings? (How many Kempe changes are needed.)

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Conjecture

Any pair of k-colourings of a graph of maximum degree k on n vertices are joined by a sequence of $O(n^2)$ Kempe changes.

Thank You

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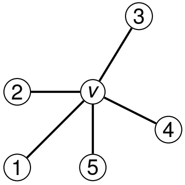

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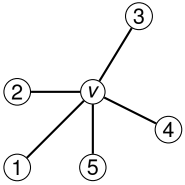

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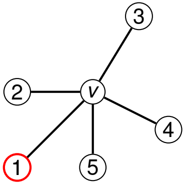

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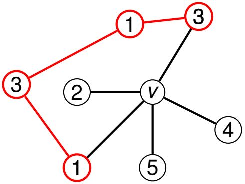

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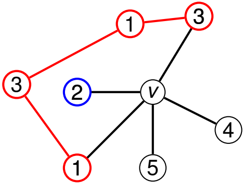

Else remove colour 2 from the neighbours of v with a Kempe change of a (2,4)-component.

k-Regular Graphs that are not 3-connected

If G is not 3-connected it has a cutset C of size 1 or 2. If this is a clique, then apply the clique cutset lemma (and then notice that the union of C and each connected component of $G - C$ is $(k-1)$-degenerate).

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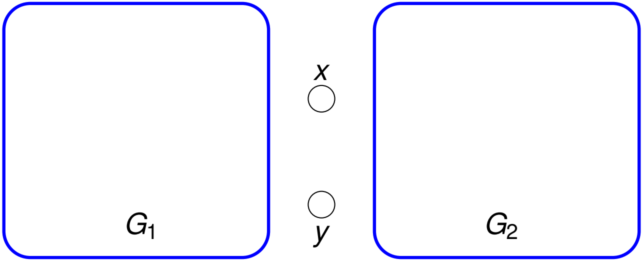

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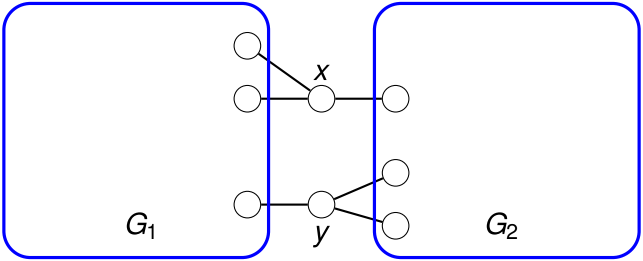

We can assume that x has more than one neighbour in G_1 and y has more than one neighbour in G_2.

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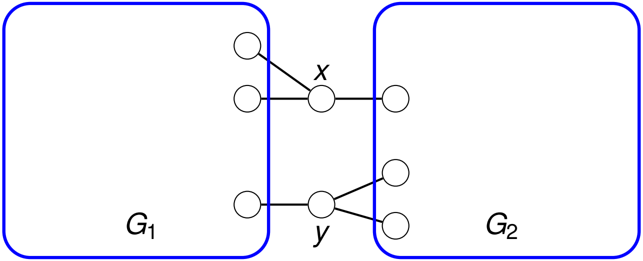

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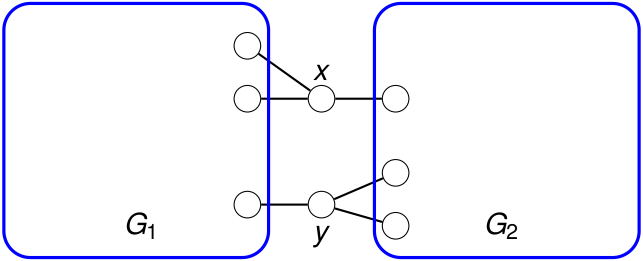

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Just to need that when x and y are coloured alike we can apply Kempe changes until they differ.

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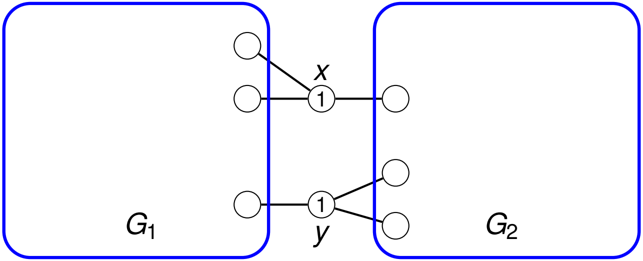

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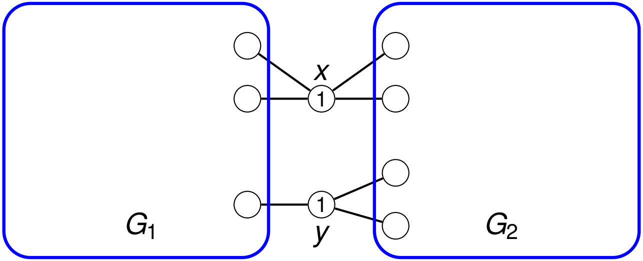

Suppose x has two neighbours in each of G_1 and G_2.

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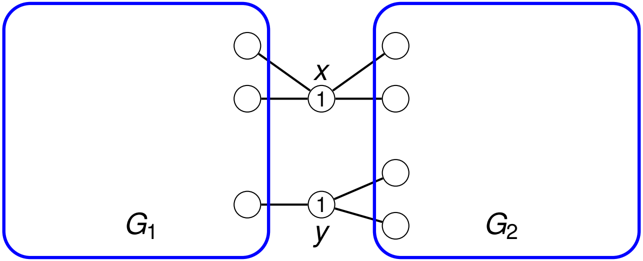

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Then x has no neighbour in G_1 coloured 3.
And x has no neighbour in G_2 coloured 2.

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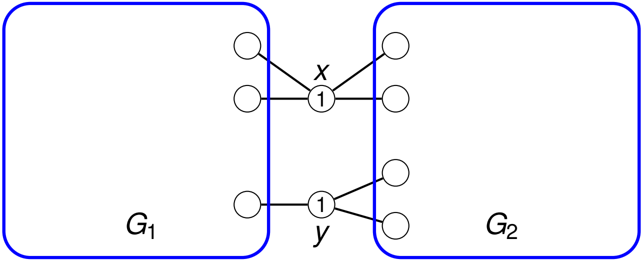

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Kempe change $(2, 3)$-components in G_1 so that x has no neighbour coloured 2.

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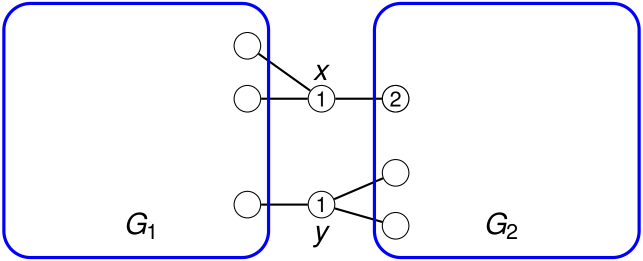

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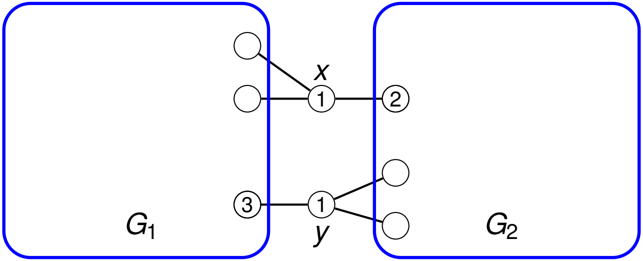

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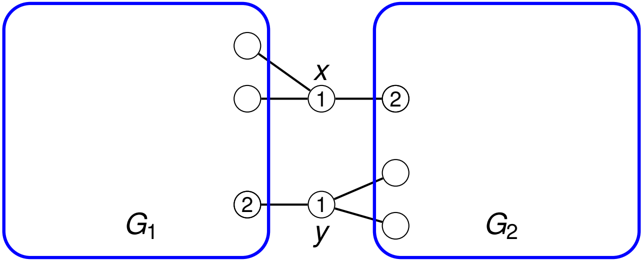

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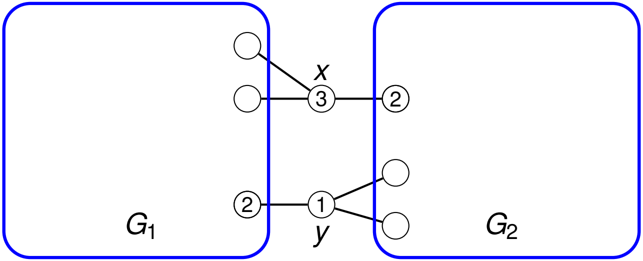

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