Reconfiguring Vertex Colourings of 2-Trees

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The k-colouring graph

A proper k-vertex-colouring of a graph H is a function $f : V(H) \to \{1, 2, \ldots, k\}$ such that $f(x) \neq f(y)$ for all $xy \in E(H)$. Henceforth we call these k-colourings, since we are concerned only with proper k-vertex-colourings.

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Cereceda, van den Heuvel, and Johnson (2008) prove that $G_k(H)$ is connected for all $k \geq \mathrm{col}(H) + 1$ (where $\mathrm{col}(H)$ is the colouring number of H, defined as $\mathrm{col}(H) = \max\{\delta(G) \mid G \subseteq H\} + 1$).

Hamiltonicity of the k-colouring graph

Choo and MacGillivray (2011) prove that for any graph H there is a least integer $k_0(H)$ such that $G_k(H)$ has a Hamilton cycle for all $k \geq k_0(H)$. They call $k_0(H)$ the Gray code number of H, and prove that $k_0(H) \leq \text{col}(H) + 2$.

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- **Complete Graphs.** $k_0(K_1) = 3$ $(= \text{col}(K_1) + 2)$ and $k_0(K_n) = n + 1$ $(= \text{col}(K_n) + 1)$ for all $n \geq 2$.

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- **Trees.** $k_0(T) = 4$ $(= \mathrm{col}(T) + 2)$ unless T is a star with an odd number of vertices greater than one, in which case $k_0(T) = 3$ $(= \mathrm{col}(T) + 1)$.

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Celaya, Choo, MacGillivray and Seyffarth (2016) prove that

- **Complete Bipartite Graphs.** $k_0(K_{\ell,r}) = 3$ when ℓ and r are both odd, and $k_0(K_{\ell,r}) = 4$ otherwise.

Gray code numbers for 2-trees

Theorem (Cavers, KS)

If H is a 2-tree, then $k_0(H) = 4$ unless $H \cong T \vee \{u\}$ for some tree T and vertex u, where T is a star with an odd number of vertices greater than one, or the bipartition of $V(T)$ has two even parts; in these cases, $k_0(H) = 5$.

Naive Approach

- Proof by induction; base case K_3.

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- For $j = 0, 1, \ldots, N-1$, let $F_j \subseteq V(G_4(H))$ be the set of 4-colouring of H that agree with f_j on $V(H')$; then $\{F_0, F_1, \ldots, F_{N-1}\}$ is a partition of the vertices of $G_4(H)$.

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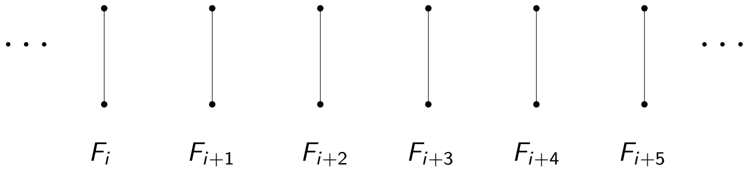
- $H[F_j] \cong K_2$ for each j, $0 \leq j \leq N-1$.

Naive Approach



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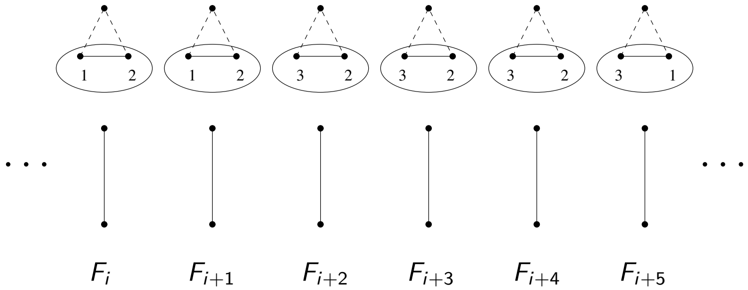

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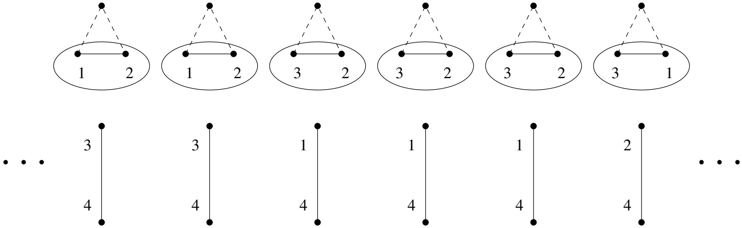
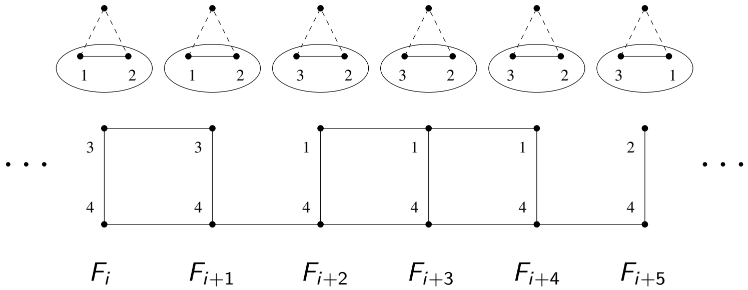


F_i F_{i+1} F_{i+2} F_{i+3} F_{i+4} F_{i+5}

Naive Approach



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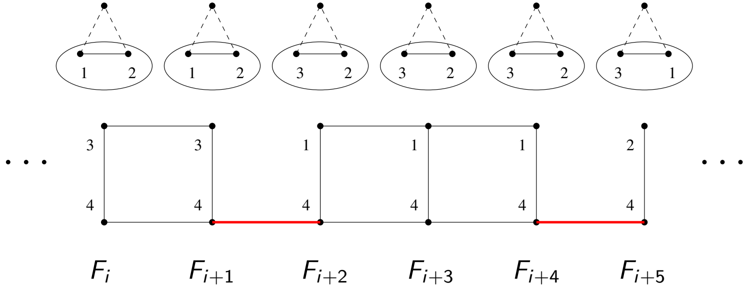
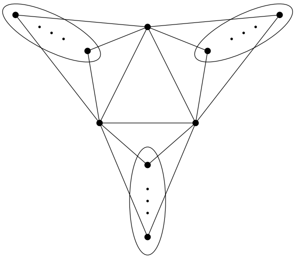
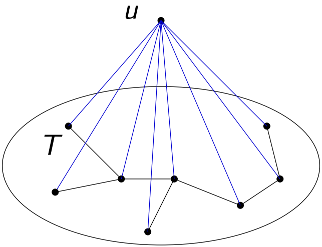
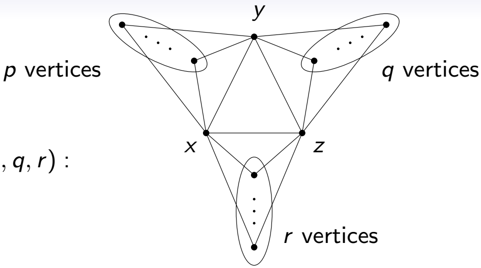


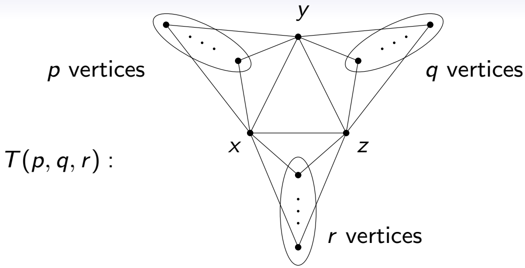
2-trees of diameter two



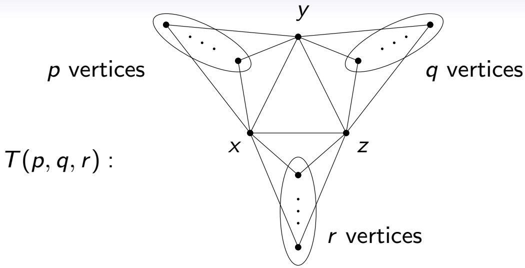


$T(p,q,r)$:





- Start with the 4-colouring graph of K_3, and let $V(G_4(K_3)) = \{f_0, f_1, \ldots, f_{N-1}\}$. Let $f_0 f_1 f_2 \ldots f_{N-1}$ be a hamilton path in $G_4(K_3)$.



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- Let F_i denote the set of 4-colourings of $T(p, q, r)$ that agree with f_i on $\{x, y, z\}$. Then $\{F_0, F_1, \ldots, F_{N-1}\}$ is a partition of the vertex set of $G = G_4(T(p, q, r))$.

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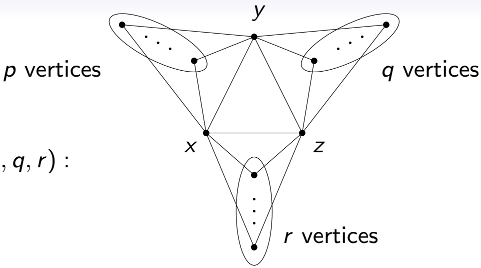

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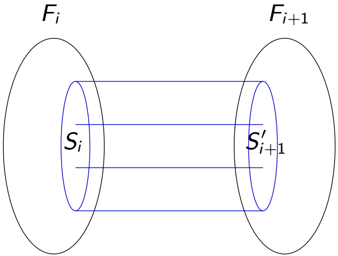
- The subgraph induced by F_i is isomorphic to Q_{p+q+r}.



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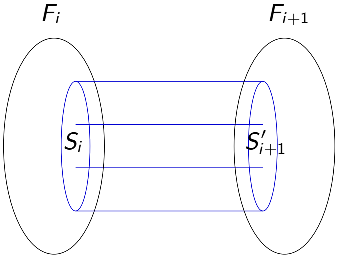
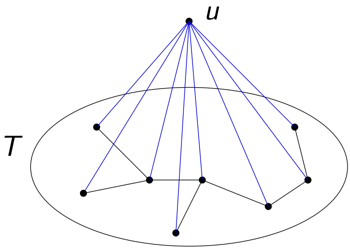


- The subgraph induced by F_i is isomorphic to Q_{p+q+r}.
- Let $S_i \subseteq F_i$ and $S'_{i+1} \subseteq F_{i+1}$ denote the vertices incident to the edges of $[F_i, F_{i+1}]$. Then $G[S_i]$ and $G[S'_{i+1}]$ are both isomorphic to one of Q_p, Q_q or Q_r, and $G[S_i \cup S_{i+1}]$ is isomorphic to one of Q_{p+1}, Q_{q+1} or Q_{r+1}, respectively.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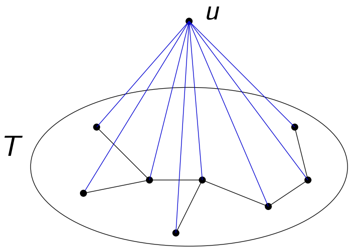

Lemma

Let T be a tree on at least three vertices. Then $G_4(T \vee \{u\})$ has a Hamilton cycle unless T is a star with at least three vertices, or the bipartition of $V(T)$ has two even parts.

The Lemma with the really long horrible proof!

Lemma

Let T be a tree with bipartition (A, B) where $|A| = \ell$ and $|B| = r$, and let $G_3(T)$ be the 3-colouring graph of T with colours $C = \{1, 2, 3\}$. Define c_{ij} to be the vertex of $G_3(T)$ with $c_{ij}(a) = i$ for all $a \in A$ and $c_{ij}(b) = j$ for all $b \in B$.

- If $\ell, r > 0$ are both even, then $G_3(T)$ has no spanning subgraph consisting only of paths whose ends are in $\{c_{12}, c_{13}, c_{21}, c_{23}, c_{31}, c_{32}\}$.
- If $\ell > 1$ is odd and $r > 0$ is even, then $G_3(T)$ has a hamilton path from c_{12} to c_{13}.
- If $\ell > 1$ and $r > 1$ are both odd, then $G_3(T)$ has a hamilton path from c_{12} to c_{23}.

Constructing 2-trees of diameter at least three

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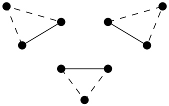
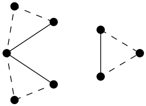
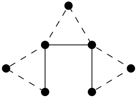


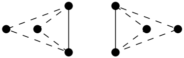
Operation I



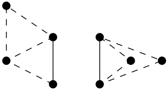
Operation II



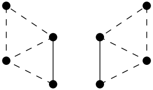
Operation III



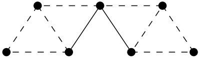
Operation IV



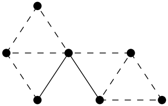
Operation V



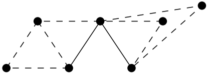
Operation VI



Operation VII



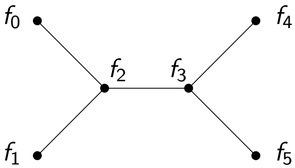
Operation VIII



Operation IX

Main idea of the proof

- Let H be a 2-tree, and let H' be a 2-tree obtained from H by applying one of the operations I through IX.
- Let $V(G_4(H)) = \{f_0, f_1, \ldots, f_{N-1}\}$ and let $F_j \subseteq V(G_4(H'))$ be the set of 4-colourings of H' that agree with f_j of the vertices of H.
- Let T be a spanning tree of maximum degree at most four of $G_4[H]$ (such a spanning tree exists).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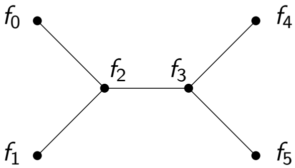
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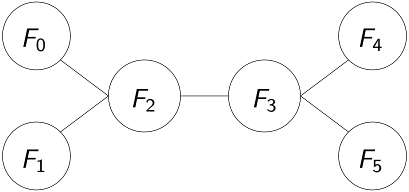
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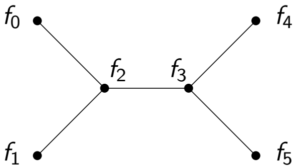
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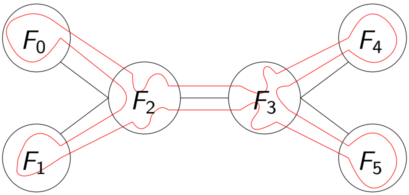
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References

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- K. Choo and G. MacGillivray, *Gray code numbers for graphs*, Ars Math. Contemp. **4** (2011), 125–139.

What's next?

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- Gray code numbers of 3-trees?

What's next?

- Gray code numbers of 3-trees?
- Gray code numbers of k-trees?

What's next?

- Gray code numbers of 3-trees?
- Gray code numbers of k-trees?
- Gray code numbers of chordal graphs?

Thank you!